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(22) ★ Signed Measure :- Let $\mathcal{A}$ be a σ -algebra of subset of X . Then an extended real valued set function $\nu: \mathcal{A} \rightarrow [-\infty, \infty]$ is called a signed measure on a measurable space $(X, \mathcal{A})$ if it satisfy the following conditions :

- (i) ν takes atleast one of the value $-\infty$ and $+\infty$
- (ii) $\nu(\emptyset) = 0$
- (iii) ν is countable additive for any sequence $\{A_i\}$ of disjoint measurable sets.
i.e. $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$

Remark : • A measure is a particular case of signed measure in which measure takes only positive values.

- Every measure is always a signed measure but the converse is not true in general.

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★ Positive Set :- A subset E of a set X is said to be positive set relative to a signed measure ν defined on a measurable space $(X, \mathcal{A})$ if

- (i) E is measurable.
- (ii) $\forall A \in E$ such that A is measurable $\Rightarrow \nu(A) \geq 0$

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* Negative Set :- A subset E of a set X is said to be negative set relative to a signed measure ν defined on a measurable space $(X, \mathcal{A})$ if

- (i) E is measurable.
- (ii) $\forall A \in \mathcal{A}$ such that A is measurable
 $\Rightarrow \nu(A) \leq 0$

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* Null Set :- A subset E of a set X is said to be null set relative to a signed measure ν defined on a measurable space $(X, \mathcal{A})$ if E is both positive and negative set.

OR

A measurable set is null set iff every measurable subset of it has signed measure zero.

* * * * *

Theorem :- Every measurable subset of a positive set is itself positive and countable union of positive sets is positive. * *

Proof :-

Let $\{A_i\}$ be a seqⁿ of positive sets in X .
 and let $A = \bigcup_{i=1}^{\infty} A_i$ — (i)

Let B be any measurable subset of A

Then we have to show that A is positive set.
 (For countable union)

To prove - B is positive set follow step (ii) only.

(i) Since A_i is positive set.

So by definition of positive set,

$\Rightarrow A_i$ is measurable set.

$\Rightarrow \bigcup_{i=1}^{\infty} A_i$ is also measurable.

[$\because$ Countable union of mble sets is also mble]

$\Rightarrow A$ is measurable set.

(ii) Consider that B be any measurable subset of A .

Write $B_n = B \cap A_n \cap A_{n-1}' \cap A_{n-2}' \cap \dots \cap A_1'$.

here $'$ denotes the complement of respective sets.

i.e. $A_n' = X - A_n$.

We know that the complement of measurable set is also measurable.

And countable intersection of measurable sets is also measurable.

$\Rightarrow B_n$ is measurable set.

Since $B_n \subset A_n$ and A_n is positive set

$\Rightarrow B_n$ is also positive set

i.e. $\nu(B_n) \geq 0$

Let $B = \bigcup_{n=1}^{\infty} B_n$

$\therefore \nu(B) = \nu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \nu(B_n) \geq 0$

$\Rightarrow \nu(B) \geq 0$.

We have (i) A is measurable set

(ii) $B \subset A$ such that B is measurable $\Rightarrow \nu(B) \geq 0$

Hence $A = \bigcup_{i=1}^{\infty} A_i$ is positive set.

Theorem: Every ^{measurable} subset of a negative set is negative itself and countable union of negative sets is negative.

Proof:

Suppose $\{A_i\}$ be a sequence of negative sets in X and let $A = \bigcup_{i=1}^{\infty} A_i$.

Let B be any measurable subset of A .

Then we have to show that B is negative set.

Consider $B_i = B \cap A_i \cap A_{i-1}' \cap A_{i-2}' \cap \dots \cap A_1'$

Here $'$ denotes the complement of respective sets.

i.e. $A_i' = X - A_i$

We know that the complement of measurable set is also measurable.

And countable intersection of measurable sets is also measurable.

$\Rightarrow B_i$ is measurable set.

— (i)

Since $B_i \subset A_i$ and A_i is negative set
 $\Rightarrow B_i$ is also negative set.

i.e. $v(B_i) \leq 0$

— (ii)

Let $B = \bigcup_{i=1}^{\infty} B_i$

$\therefore v(B) = v\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} v(B_i) \leq 0$

$\Rightarrow v(B) \leq 0$

— (iii)

Since $B = \bigcup_{i=1}^{\infty} B_i$ and $\forall B_i$ is measurable sets.
 $\Rightarrow B$ is measurable set.

— (iv)

By eqn (i), (ii) and (iv), we have

(1) B is measurable set

(2) $\forall B_i \subset B$ such that B_i is measurable $\Rightarrow \nu(B_i) \leq 0$

Thus B is negative set.

Next we prove that A is negative set.

Since A_i is negative set.

$\Rightarrow A_i$ is measurable set. [by definition]

$\Rightarrow \bigcup_{i=1}^{\infty} A_i$ is also measurable set.

$\Rightarrow A$ is measurable set — (v)

By eqn (v), (iv), (iii), we have

(1) A is measurable set.

(2) $B \subset A$ such that B is measurable $\Rightarrow \nu(B) \leq 0$

Hence A is negative set.

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Theorem: If E is a measurable set of a finite positive measure i.e. $0 < \nu(E) < \infty$. Then there is a positive set A contained in E with $\nu(A) > 0$

Proof:

Let E is a measurable set of finite positive measure i.e. $0 < \nu(E) < \infty$.

• If E itself is a positive set.

And if we take $A = E$

Then the theorem is trivially true.

• If E itself is not a positive set.

Then E must contain a subset of negative measure.
 $\Rightarrow \exists$ a measurable set $E_1 \subset E$ and a smallest positive integer n_1 such that
$$\nu(E_1) < -\frac{1}{n_1}$$

So $E = (E - E_1) \cup E_1$ and $(E - E_1) \cap E_1 = \emptyset$
 $\therefore \nu(E) = \nu(E - E_1) + \nu(E_1)$ [by countable additivity]
 $\Rightarrow \nu(E - E_1) = \nu(E) - \nu(E_1)$ — (i)

Since $\nu(E) < \infty$ and $E_1 \subset E$
 $\Rightarrow \nu(E_1) < \infty$

By eqn (i),
 $\Rightarrow \nu(E - E_1) = \nu(E) - \nu(E_1) < \infty$
 $\Rightarrow \nu(E - E_1) < \infty$
i.e. $\nu(E - E_1)$ is finite.

Given $\nu(E) > 0$ i.e. E is positive.
and by our assumption E_1 is negative.
 $\Rightarrow \nu(E - E_1)$ is positive. $[\because \nu(E - E_1) = \nu(E) - \nu(E_1) = (+ve) - (-ve) = +ve]$

If we take $A = E - E_1$,

It follows that A is positive.
i.e. $\nu(A) > 0$.

Otherwise, $E - E_1$ contains a subset of negative measure.

$\Rightarrow \exists$ a measurable set $E_2 \subset E - E_1$ and a smallest positive integer n_2 such that

$$\nu(E_2) < -\frac{1}{n_2}$$

$$\text{So } E = (E - (E_1 \cup E_2)) \cup (E_1 \cup E_2)$$

$$\therefore \nu(E) = \nu[E - (E_1 \cup E_2)] + \nu(E_1 \cup E_2)$$

$$\Rightarrow \nu[E - (E_1 \cup E_2)] = \nu(E) - \nu(E_1) - \nu(E_2) \quad \text{--- (ii)}$$

Given $\nu(E) < \infty \Rightarrow \nu(E_1) < \infty$ and $\nu(E_2) < \infty$
 $\Rightarrow \nu[E - (E_1 \cup E_2)] < \infty$

Also given $\nu(E) > 0$, $-\nu(E_1) > \frac{1}{n_1}$ & $-\nu(E_2) > \frac{1}{n_2}$
 $\Rightarrow \nu[E - (E_1 \cup E_2)] > 0$ [by (ii)]
 i.e. $E - (E_1 \cup E_2)$ is positive set.

Again we take $A = E - (E_1 \cup E_2)$

It follows that A is positive i.e. $\nu(A) > 0$

Otherwise, we repeat the same process.

So by continuation of the process, we shall get either a positive set A of E such that $\nu(A) > 0$ or a sequence $\{n_k\}$ of positive integers and a sequence $\{E_k\}$ of disjoint measurable sets such that

$$\begin{aligned} \nu(E_k) &< -\frac{1}{n_k} \\ \sum_{k=1}^{\infty} \nu(E_k) &< -\sum_{k=1}^{\infty} \frac{1}{n_k} \end{aligned} \quad \text{--- (iii)}$$

In the later case, suppose that

$$A = E - \bigcup_{k=1}^{\infty} E_k$$

$$\therefore \nu(A) = \nu\left(E - \bigcup_{k=1}^{\infty} E_k\right)$$

$$\Rightarrow \nu(A) = \nu(E) - \sum_{k=1}^{\infty} \nu(E_k)$$

$$\Rightarrow \nu(A) < \nu(E) + \sum_{k=1}^{\infty} \frac{1}{n_k} \quad \text{--- (iv)} \quad [\text{by (iii)}]$$

Since $\nu(E) < \infty$ and the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent, so it is also finite.

So from eqn (iv), $\nu(A)$ is also finite.

$$\times \left\{ \begin{array}{l} \text{Given } \nu(E) > 0 \text{ and } \sum_{k=1}^{\infty} \frac{1}{n_k} \text{ is positive} \\ \Rightarrow \nu(A) > 0 \end{array} \right\}$$

Now we have to show that A is positive.

$$\text{We have } A = E - \bigcup_{k=1}^{\infty} E_k$$

Since E is measurable set and $\{E_k\}$ is seqⁿ of disjoint measurable sets.

We know that the difference of measurable sets is also measurable.

$\Rightarrow A$ is measurable.

Let B be measurable subset of A .

$$\text{Then } B \subset A = E - \bigcup_{k=1}^{\infty} E_k$$

$$\Rightarrow B \subset E - \bigcup_{k=1}^{\infty} E_k$$

So the choice of positive integer $\nu(B) \geq \frac{1}{n_i - 1}$ --- (v)

This is true $\forall i \in \mathbb{N}$ and $n_i \rightarrow \infty$ in (v),

we get $\nu(B) \geq 0$

Hence A is positive set with $\nu(A) > 0$.

Theorem: If E is measurable set with finite negative measure i.e. $-\infty < \nu(E) < 0$. Then E contains a negative set A with $\nu(A) < 0$.

Proof:

- If E itself is negative set.
And if we take $A = E$
Then the theorem is trivially true.
- If E itself is not a negative set.

Then E must contain a subset of positive measure
 $\Rightarrow$ $\exists$ a measurable set $E_1 \subset E$ and a smallest positive integer n_1 such that

$$\nu(E_1) > \frac{1}{n_1}$$

$$\begin{aligned} \text{So } E &= (E - E_1) \cup E_1 & \text{f } (E - E_1) \cap E_1 &= \emptyset \\ \therefore \nu(E) &= \nu(E - E_1) + \nu(E_1) \\ \Rightarrow \nu(E - E_1) &= \nu(E) - \nu(E_1) \end{aligned} \quad \text{--- (i)}$$

$$\begin{aligned} \text{Since } \nu(E) &> -\infty \text{ and } E_1 \subset E \\ \Rightarrow \nu(E_1) &> -\infty \end{aligned}$$

So by eqn (i),

$$\begin{aligned} \nu(E - E_1) &= \nu(E) - \nu(E_1) > -\infty \\ \Rightarrow \nu(E - E_1) &> -\infty \end{aligned}$$

Given $\nu(E) < 0$ i.e. E is negative
 and by our assumption E_1 is positive
 $\Rightarrow \nu(E - E_1) < 0$ [$\because \nu(E - E_1) = \nu(E) - \nu(E_1) = (-ve) - (+ve) = (-ve)$]
 i.e. $(E - E_1)$ is negative.

If we take $A = E - E_1$

It follows that A is negative

i.e. $\nu(A) < 0$

Otherwise, $E - E_1$ contains a subset of positive measure

$\Rightarrow \exists$ a measurable set $E_2 \subset E - E_1$ and smallest positive integer n_2 such that

$$\nu(E_2) > \frac{1}{n_2}$$

So $E = [E - (E_1 \cup E_2)] \cup (E_1 \cup E_2)$ & $[E - (E_1 \cup E_2)] \cap (E_1 \cup E_2) = \emptyset$

$\therefore \nu(E) = \nu[E - (E_1 \cup E_2)] + \nu(E_1 \cup E_2)$

$\Rightarrow \nu[E - (E_1 \cup E_2)] = \nu(E) - \nu(E_1 \cup E_2)$ — (ii)

Given $\nu(E) > -\infty \Rightarrow \nu(E_1) > -\infty$ and $\nu(E_2) > -\infty$

$\Rightarrow \nu[E - (E_1 \cup E_2)] > -\infty$ [by (ii)]

Also given $\nu(E) < 0$, $\nu(E_1) > \frac{1}{n_1}$ & $\nu(E_2) > \frac{1}{n_2}$

$\Rightarrow \nu[E - (E_1 \cup E_2)] < 0$ [by (ii)]

i.e. $E - (E_1 \cup E_2)$ is negative set.

Again we take $A = E - (E_1 \cup E_2)$

It follows that A is negative with $\nu(A) < 0$

Otherwise, we repeat the same process.

So by continuation of the process, we shall get either a negative set A of E such that $\nu(A) < 0$ or a seqⁿ $\{n_k\}$ of positive integers and a seqⁿ

$\{E_k\}$ of disjoint measurable sets such that
 $\nu(E_k) > \frac{1}{n_k}$ — (iii)

In the later case, suppose that

$$A = E - \bigcup_{k=1}^{\infty} E_k$$

$$\therefore \nu(A) = \nu(E) - \sum_{k=1}^{\infty} \nu(E_k)$$

$$\Rightarrow \nu(A) < \nu(E) - \sum_{k=1}^{\infty} \frac{1}{n_k} \quad \text{--- (iv)} \quad [\text{by (iii)}]$$

Since $\nu(E) > -\infty$ and the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent, so it is also finite.

From eqn (iv), $\nu(A)$ must be finite.

Now we prove that A is negative.

$$\text{We have } A = E - \bigcup_{k=1}^{\infty} E_k$$

Since E is measurable set and $\{E_k\}$ is seqn of disjoint measurable sets.

We know that the difference of measurable sets is also measurable.

$\Rightarrow A$ is measurable.

Let B be measurable subset of A .

$$\text{Then } B \subset A = E - \bigcup_{k=1}^{\infty} E_k$$

$$\Rightarrow B \subset E - \bigcup_{k=1}^{\infty} E_k$$

So the choice of positive integer $\nu(B) \leq \frac{1}{n_{i-1}}$ — (v)

This is true $\forall i \in \mathbb{N}$ and $n_i \rightarrow \infty$ in (v), we get

$$\nu(B) \leq 0$$

Hence A is negative set with $\nu(A) \leq 0$.

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(273) ★. Hahn Decomposition Theorem :- Let ν be a signed measure on the measurable space $(X, \mathcal{A})$, where $\mathcal{A}$ being σ -algebra of subsets of X . Then there is a positive set P and a negative set Q such that

$$X = P \cup Q \quad \text{and} \quad P \cap Q = \emptyset$$

Proof :

Let $\mathcal{A}$ be a σ -algebra of subsets of X . and let ν be signed measure on a measurable space $(X, \mathcal{A})$.

Since ν assumes atmost one of the values $-\infty, +\infty$. So without loss of generality, we may assume that ν does not take the value $-\infty$.

Consider the family β of all negative subsets of X . and let $\lambda = \inf \{ \nu(E) : E \in \beta \}$ — (i)

Then $\exists$ a sequence $\{E_n\}$ in β such that

$$\lim_{n \rightarrow \infty} \nu(E_n) = \lambda$$

— (ii)

Since β is a family of negative sets.

$\Rightarrow \{E_n\}$ is a sequence of negative sets.

$\Rightarrow \bigcup_{n=1}^{\infty} E_n$ is also negative.

$\Rightarrow Q$ is a negative set on taking $Q = \bigcup_{n=1}^{\infty} E_n$.

Thus Q is negative subset of X and by eqn (i),

$$\lambda \leq \nu(Q)$$

— (iii)

Now consider $(Q - E_n) \subset Q$

$$\text{and } v(Q - E_n) \leq 0$$

[$\because Q$ is negative set]

$$\therefore Q = (Q - E_n) \cup E_n \quad \text{and} \quad (Q - E_n) \cap E_n = \emptyset$$

$$\Rightarrow v(Q) = v(Q - E_n) + v(E_n) \leq v(E_n)$$

$$\Rightarrow v(Q) \leq v(E_n) \quad \forall n \in \mathbb{N} \quad \text{and} \quad E_n \in \beta$$

In the above inequality, take limit $n \rightarrow \infty$,

$$v(Q) \leq \lim_{n \rightarrow \infty} v(E_n)$$

$$\Rightarrow v(Q) \leq \lambda$$

— (iv)

[by eqn (ii)]

By eqn (iii) & (iv),

$$v(Q) = \lambda$$

— (v)

$$\Rightarrow \lambda > -\infty$$

Next our aim to show that $P = X - Q$ is a positive set of X .

Suppose on contrary, P is negative subset of X .

By definition of negative set,

For every measurable set $E \subset P$ with $v(E) \leq 0$.

* { Now, E is a measurable subset of P }

* { If E is a measurable set with finite negative measure. }

Then by the result, "If E is measurable set with finite negative measure i.e. $-\infty < v(E) < 0$. Then E contains a negative set A with $v(A) < 0$ ".

$\Rightarrow E$ contains a negative set A with $v(A) < 0$

Since A and Q both are disjoint negative subsets of X .

$\Rightarrow A \cup Q$ is also negative.

[$\because$ The countable union of negative sets is negative]

$\Rightarrow A \cup Q \in \beta$

So by eqn (i),

$$\lambda \leq v(A \cup Q)$$

$$= v(A) + v(Q)$$

$$\Rightarrow \lambda \leq v(A) + \lambda$$

[by eqn (v)]

$$\Rightarrow v(A) \geq 0$$

which is contradict the fact that $v(A) < 0$

Thus our assumption that P is negative set, is wrong.

$\Rightarrow P$ is positive set.

Hence $P = X - Q$ is positive set & Q is negative set such that

$$X = P \cup Q \quad \text{and} \quad P \cap Q = \emptyset.$$

- A decomposition of a measurable space X into two subsets P and Q such that $X = P \cup Q$ and $P \cap Q = \emptyset$, where P and Q are positive and negative sets respectively, relative to signed measure v is called Hahn decomposition of the signed measure v . Also P and Q are called positive & negative components of X .

Remark: • A Hahn decomposition is not unique.

- If $\{P, Q\}$ is a Hahn decomposition for ν , Then we may define two signed measures ν^+ & ν^- such that $\nu = \nu^+ - \nu^-$ by setting

$$\nu^+(E) = \nu(E \cap P)$$

$$\text{and } \nu^-(E) = -\nu(E \cap Q)$$

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* Singular Measure :- Two measures ν_1 and ν_2 on a measurable space $(X, \mathcal{A})$ are said to be mutually singular or singular if there are disjoint measurable sets A and B with $X = A \cup B$ such that

$$\nu_1(A) = \nu_2(B) = 0$$

Symbolically it is written as $\nu_1 \perp \nu_2$.

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* Jordan decomposition :- Let ν be a signed measure on the measurable space $(X, \mathcal{A})$. If there are two mutually singular measures ν_1 and ν_2 on $(X, \mathcal{A})$ such that

$$\nu = \nu_1 - \nu_2$$

Then the decomposition of ν is said to be Jordan decomposition.

Note: • Since ν assumes at most one of the values $+\infty$ and $-\infty$,

Therefore either ν_1 or ν_2 must be finite.

(274) * Absolute value or Total variation :- The measure $|v|$ defined

by $|v|(E) = v^+(E) + v^-(E)$
is called the absolute value or total variation
of v .

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* Absolutely Continuous Measure :- Let v and μ
be two measure
functions on a measurable space $(X, \mathcal{A})$. Then
a measure v is said to be absolutely continuous
measure with respect to the measure μ if
 $v(A) = 0$ for each set A , for which $\mu(A) = 0$.
i.e. $\mu(A) = 0 \quad \forall A \in \mathcal{A} \Rightarrow v(A) = 0$

Symbolically it is written as $v \ll \mu$.

Note :- • If μ is σ -finite

Then $v \ll \mu$ also $\mu \ll v$

i.e. $\mu(A) = 0 \quad \forall A \in \mathcal{A} \Leftrightarrow v(A) = 0$

• In case of signed measure, we say that

(1.) $v \ll \mu$ if $|v| \ll |\mu|$

(2.) $v \perp \mu$ if $|v| \perp |\mu|$

* σ -Finite Measure :- A signed measure v on
measurable space $(X, \mathcal{A})$ is
said to be σ -finite if $X = \bigcup_{i=1}^{\infty} X_i$ such that

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For all $X_i \in \mathcal{A}$ and $v(X_i) < \infty \quad \forall i$.

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(276) ★. Radon Nikodym Theorem :- Let $(X, \mathcal{B}, \mu)$ be a σ -finite measure

and let v be a measure defined on $\mathcal{B}$ which is absolutely continuous with respect to μ .

Then there is a non-negative measurable function f such that for each set E in $\mathcal{B}$, we have

$$v(E) = \int_E f \, d\mu$$

The function f is unique in the sense that if g is any measurable function with this property, then $g=f$ a.e. [4].

Proof :

First we prove this theorem for finite measure. Then the extension of finite to σ -finite case is easy to prove.

Let μ be a finite measure.

Then $v - \alpha\mu$ be a signed measure for rational number α .

Let (A_α, B_α) be a Hahn decomposition for α ; for a signed measure $(v - \alpha\mu)$.

Take $X = A_\alpha$ & $B_\alpha = \emptyset$

Then $X = A_\alpha \cup B_\alpha$ and $A_\alpha \cap B_\alpha = \emptyset$

Again, let (A_β, B_β) be another Hahn decomposition for β ; for a signed measure $(v - \beta\mu)$.

Then $X = A_\beta \cup B_\beta$ and $A_\beta \cap B_\beta = \emptyset$

$$\text{Now } B_\alpha - B_\beta = B_\alpha \cap B_\beta^c$$

$$\Rightarrow B_\alpha - B_\beta = B_\alpha \cap A_\beta$$

So there are two cases :

Case I. $B_\alpha - B_\beta \subset B_\alpha$

Since B_α is a negative set with respect to $(\nu - \alpha\mu)$.

$$\therefore (\nu - \alpha\mu)(B_\alpha - B_\beta) \leq 0$$

$$\Rightarrow \nu(B_\alpha - B_\beta) - \alpha\mu(B_\alpha - B_\beta) \leq 0 \quad \text{--- (i)}$$

Case II. $B_\alpha - B_\beta \subset A_\beta$

Since A_β is a positive set with respect to $(\nu - \beta\mu)$

$$\therefore (\nu - \beta\mu)(B_\alpha - B_\beta) \geq 0$$

$$\Rightarrow \nu(B_\alpha - B_\beta) - \beta\mu(B_\alpha - B_\beta) \geq 0$$

$$\Rightarrow -\nu(B_\alpha - B_\beta) + \beta\mu(B_\alpha - B_\beta) \leq 0 \quad \text{--- (ii)}$$

By addition of (i) & (ii), we get

$$\Rightarrow (\beta - \alpha)\mu(B_\alpha - B_\beta) \leq 0$$

$$\Rightarrow \mu(B_\alpha - B_\beta) \leq 0, \text{ for } \beta > \alpha \quad \text{--- (iii)}$$

But in general $\mu(B_\alpha - B_\beta) \geq 0$ [$\mu = \text{measure}$] --- (iv)

From eqn (iii) & (iv), we have

$$\mu(B_\alpha - B_\beta) = 0$$

Now we use the following lemma,

"Suppose that for each α in a dense set D of real numbers, there is assigned a set

$B_\alpha \in \beta$ s.t. $\mu(B_\alpha - B_\beta) = 0$ for $\alpha < \beta$.
 Then there is a measurable function f such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $(X - B_\alpha) = A_\alpha$. If g is any other function with this property, then $g = f$ a.e."

Then by above lemma, we have a unique measurable function f such that

$$f \leq \alpha \text{ a.e. on } B_\alpha$$

$$\text{and } f \geq \alpha \text{ a.e. on } X - B_\alpha = A_\alpha.$$

Since $A_\alpha = X$ and $B_\alpha = \emptyset$

Therefore we may take f is non-negative function.

Let E be arbitrary set in β and we define

$$E_k = E \cap (B_{\frac{k+1}{N}} - B_{\frac{k}{N}})$$

$$\text{and } E_\infty = E - \left(\bigcup_{k=0}^{\infty} B_{\frac{k}{N}} \right)$$

$$\text{Then } E = E_\infty \cup \left(\bigcup_{k=0}^{\infty} E_k \right) \quad \text{--- (v)}$$

and this is a disjoint union of measurable sets.

$$\therefore \nu(E) = \nu(E_\infty) + \nu\left(\bigcup_{k=0}^{\infty} E_k\right)$$

$$\Rightarrow \nu(E) = \nu(E_\infty) + \sum_{k=0}^{\infty} \nu(E_k) \quad \text{--- (vi)}$$

$$\text{Since } E_k \subset B_{\frac{k+1}{N}} - B_{\frac{k}{N}} = B_{\frac{k+1}{N}} \cap B_{\frac{k}{N}}^c$$

$$\Rightarrow E_k \subset B_{\frac{k+1}{N}} \cap A_{\frac{k}{N}}$$

$$\therefore \frac{k}{N} \leq f(x) \leq \frac{k+1}{N} \quad \text{on } x \in E_k \quad \text{--- (vii)}$$

$$\Rightarrow \int_{E_k} \frac{k}{N} d\mu \leq \int_{E_k} f(x) d\mu \leq \int_{E_k} \frac{k+1}{N} d\mu \quad \text{--- (viii)}$$

$$\Rightarrow \frac{K}{N} \mu(E_K) \leq \int_{E_K} f \, d\mu \leq \frac{K+1}{N} \mu(E_K) \quad \text{--- (viii)}$$

Also, $E_K \subset A_{\frac{K}{N}}$, which is a positive set.

$$\therefore \left(\nu - \frac{K}{N} \mu \right) (E_K) \geq 0$$

$$\Rightarrow \nu(E_K) - \frac{K}{N} \mu(E_K) \geq 0$$

$$\Rightarrow \nu(E_K) \geq \frac{K}{N} \mu(E_K) \quad \text{--- (ix)}$$

Also, $E_K \subset B_{\frac{K+1}{N}}$, which is a negative set.

$$\therefore \left(\nu - \frac{K+1}{N} \mu \right) (E_K) \leq 0$$

$$\Rightarrow \nu(E_K) - \frac{K+1}{N} \mu(E_K) \leq 0$$

$$\Rightarrow \nu(E_K) \leq \frac{K+1}{N} \mu(E_K) \quad \text{--- (x)}$$

From eqn (ix) & (x), we have

$$\frac{K}{N} \mu(E_K) \leq \nu(E_K) \leq \frac{K+1}{N} \mu(E_K)$$

$$\therefore \frac{K}{N} \mu(E_K) - \frac{1}{N} \mu(E_K) \leq \nu(E_K) - \frac{1}{N} \mu(E_K) \leq \frac{K+1}{N} \mu(E_K) - \frac{1}{N} \mu(E_K)$$

$$\Rightarrow \frac{K-1}{N} \mu(E_K) \leq \nu(E_K) - \frac{1}{N} \mu(E_K) \leq \frac{K}{N} \mu(E_K) \quad \text{--- (xi)}$$

Similarly, we have

$$\frac{K+1}{N} \mu(E_K) \leq \nu(E_K) + \frac{1}{N} \mu(E_K) \leq \frac{K+2}{N} \mu(E_K) \quad \text{--- (xii)}$$

So by eqn (viii), (xi) and (xii), we have

$$\nu(E_K) - \frac{1}{N} \mu(E_K) \leq \int_{E_K} f \, d\mu \leq \nu(E_K) + \frac{1}{N} \mu(E_K) \quad \text{--- (xiii)}$$

Now on E_∞ , we have $f = \infty$ a.e.

• If $\mu(E_\infty) > 0$

Then $E_\infty \subset A_\alpha$ for every rational number α .
and A_α is positive set.

$$\begin{aligned}
 & \because (v - \alpha \mu)(E_\infty) \geq 0 \\
 \Rightarrow & v(E_\infty) - \alpha \mu(E_\infty) \geq 0 \\
 \Rightarrow & v(E_\infty) \geq \alpha \mu(E_\infty) \quad \forall \alpha \\
 \Rightarrow & v(E_\infty) = \infty \quad \text{a.e.} \quad [\because v(E_\infty) \geq 0]
 \end{aligned}$$

• If $\mu(E_\infty) = 0$

Since v is absolutely continuous measure with respect to μ i.e. $v \ll \mu$.

$$\Rightarrow v(E_\infty) = 0$$

$$\text{In either case, } v(E_\infty) = \int_{E_\infty} f \, d\mu \quad \text{--- (xiv)}$$

Adding (xiii) & (xiv), we have

$$\begin{aligned}
 \sum_{k=0}^{\infty} \left[v(E_k) - \frac{1}{N} \mu(E_k) \right] & \leq \sum_{k=0}^{\infty} \int_{E_k} f \, d\mu \leq \sum_{k=0}^{\infty} \left[v(E_k) + \frac{1}{N} \mu(E_k) \right] \\
 + v(E_\infty) & \quad + \int_{E_\infty} f \, d\mu \quad + v(E_\infty)
 \end{aligned}$$

Since N is arbitrary, so take $N \rightarrow \infty$

$$\Rightarrow \sum_{k=0}^{\infty} v(E_k) + v(E_\infty) \leq \int_E f \, d\mu \leq \sum_{k=0}^{\infty} v(E_k) + v(E_\infty)$$

$$\Rightarrow v(E) \leq \int_E f \, d\mu \leq v(E) \quad [\text{by (v), (vi)}]$$

$$\Rightarrow \boxed{v(E) = \int_E f \, d\mu \quad \forall E \in \beta}$$

Thus this result is true for a finite measure μ .

To show the theorem is true for σ -finite measure μ , decompose X into a countable union of X_i i.e. $X = \bigcup_{i=0}^{\infty} X_i$, for a finite μ -measures. Applying the same argument as above for each X_i and we get the required result.

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* Radon Nikodym Derivative :- The unit function f define by the

Radon Nikodym Theorem,

$$v(E) = \int_E f \, d\mu \quad \forall E \in \beta$$

is called as Radon Nikodym derivative of v with respect to μ and it is denoted by

$$f = \left[\frac{dv}{d\mu} \right]$$

and hence

$$v(E) = \int_E \frac{dv}{d\mu} \cdot d\mu$$

Que. : If v_1 and v_2 are σ -finite measure on (X, β) and $v_1 \ll \mu$ & $v_2 \ll \mu$.

Then

$$\frac{d(v_1+v_2)}{d\mu} = \frac{dv_1}{d\mu} + \frac{dv_2}{d\mu}$$

and $\frac{dv_1}{d\mu} = - \frac{d(-v_1)}{d\mu}$

Soln :

Suppose $E \in \beta$ be arbitrary.

Since v_1 and v_2 are σ -finite measure on (X, β) .

$\therefore v_1 + v_2$ is also σ -finite measure.

Also, $v_1 \ll \mu$ and $v_2 \ll \mu$.

So by definition of absolute continuous,

$$\Rightarrow \mu(E) = 0 \Rightarrow v_1(E) = 0 \text{ and } \mu(E) = 0 \Rightarrow v_2(E) = 0$$

$$\Rightarrow \mu(E) = 0 \Rightarrow v_1(E) + v_2(E) = 0$$

$$\Rightarrow \mu(E) = 0 \Rightarrow (v_1 + v_2)(E) = 0$$

$$\therefore (v_1 + v_2) \ll \mu$$

Since we know that

$$(v_1 + v_2) \mid E = v_1(E) + v_2(E)$$

$$\Rightarrow \int_E \frac{d(v_1 + v_2)}{d\mu} \cdot d\mu = \int_E \frac{dv_1}{d\mu} \cdot d\mu + \int_E \frac{dv_2}{d\mu} \cdot d\mu$$

$$\Rightarrow \int_E \frac{d(v_1 + v_2)}{d\mu} \cdot d\mu = \int_E \left(\frac{dv_1}{d\mu} + \frac{dv_2}{d\mu} \right) d\mu$$

$$\Rightarrow \frac{d(v_1 + v_2)}{d\mu} = \frac{dv_1}{d\mu} + \frac{dv_2}{d\mu}$$

$$\text{Now, } \int_E -\frac{d(-v)}{d\mu} \cdot d\mu = -v(E)$$

$$= v(E)$$

$$\Rightarrow \int_E -\frac{d(-v)}{d\mu} = \int_E \frac{dv}{d\mu} \cdot d\mu$$

$$\Rightarrow -\frac{d(-v)}{d\mu} = \frac{dv}{d\mu}$$

Que. $\circ$ If v is a σ -finite signed measure and μ is a σ -finite measure such that $v \ll \mu$. Then show that $\frac{d|v|}{d\mu} = \left| \frac{dv}{d\mu} \right|$

Soln:

Since v is a σ -finite signed measure.

So, let $\{A, B\}$ be Hahn decomposition of v , where A is positive set & B is negative set, such that $v = v^+ - v^-$

$$\therefore |v| = |v^+ - v^-| = v^+ + v^-$$

$$\text{On the set } A, \quad \left| \frac{dv}{d\mu} \right| = \frac{dv^+}{d\mu}$$

$$\text{On the set } B, \quad \left| \frac{dv}{d\mu} \right| = \frac{dv^-}{d\mu}$$

$$\therefore \left| \frac{dv}{d\mu} \right| = \frac{dv^+}{d\mu} + \frac{dv^-}{d\mu} = \frac{d}{d\mu} (v^+ + v^-)$$

$$\Rightarrow \left| \frac{dv}{d\mu} \right| = \frac{d|v|}{d\mu}$$

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(278) ★ Lebesgue Decomposition Theorem :- Let $(X, \mathcal{B}, \mu)$ be a σ -finite

measure space and ν is a σ -finite measure defined on $\mathcal{B}$. Then we can find a measure ν_0 , singular with respect to μ and a measure ν_1 , absolutely continuous with respect to μ such that

$$\nu = \nu_0 + \nu_1$$

The measures ν_0 and ν_1 are unique.

OR

Let $(X, \mathcal{B}, \mu)$ be a σ -finite measure space and ν is a σ -finite measure defined on $\mathcal{B}$. Then $\exists$ two uniquely determined measure ν_0 and ν_1 such that $\nu = \nu_0 + \nu_1$, $\nu_0 \perp \mu$ and $\nu_1 \ll \mu$.

Proof :

$$\text{Let } \lambda = \mu + \nu$$

Since μ and ν are σ -finite measures.

$\therefore \lambda$ is also σ -finite measure.

$$\left. \begin{array}{l} \text{Evidently } \mu \ll \lambda, \mu \perp \lambda \\ \text{and } \nu \ll \lambda, \nu \perp \lambda \end{array} \right\} \text{--- (a)}$$

So, by Radon Nikodym Theorem, $\exists$ a non-negative measurable function $f, g : X \rightarrow [-\infty, \infty]$ such that

$$\begin{aligned} \mu(E) &= \int_E f \, d\lambda \\ \text{and } \nu(E) &= \int_E g \, d\lambda \end{aligned} \quad \forall E \in \mathcal{B}.$$

$$\text{Let } A = \{x \in X : f(x) > 0\} \quad \text{--- (i)}$$

$$\text{and } B = \{x \in X : f(x) = 0\} \quad \text{--- (ii)}$$

Then $X = A \cup B$, $A \cap B = \emptyset$

Furthermore, $\mu(B) = \int_B f d\lambda = 0$ — (iii)
[by (ii)]

Now we define two functions $\nu_0, \nu_1 : \mathcal{B} \rightarrow [-\infty, \infty]$
by $\nu_0(E) = \nu(E \cap B)$
and $\nu_1(E) = \nu(E \cap A)$ $\forall E \in \mathcal{B}$ — (iv)

Then ν_0 and ν_1 are measures on $\mathcal{B}$
and satisfy the condition $\nu = \nu_0 + \nu_1$

Also $\nu_0(A) = \nu(A \cap B) = \nu(\emptyset) = 0$
 $\Rightarrow \nu_0(A) = 0$ — (v)

From eqn (iii) & (iv),

$$\mu(B) = 0 = \nu_0(A) = \nu_0(X - B)$$

$$\Rightarrow \mu(B) = 0 = \nu_0(X - B)$$

$\Rightarrow \nu_0$ is mutually singular w.r. to μ
i.e. $\nu_0 \perp \mu$

Next we have to show that $\nu_1 \ll \mu$.

For this let $E \in \mathcal{B}$ be arbitrary such that
 $\mu(E) = 0$. — (vi)

$$\Rightarrow \int_E f d\lambda = 0$$

$\Rightarrow f = 0$ n.e. on E relative to λ . — (vii)

* { Also, $f(x) \geq 0 \quad \forall x \in E$ }

Since $f \geq 0$ on $E \cap A$

so by eqn (b.),

$$\nu_1(E) = \nu(E \cap A)$$

$$\leq \lambda(E \cap A)$$

[by (a.5)]

$$= 0$$

[by (vi)]

$$\Rightarrow \nu_1(E) \leq 0$$

But in general $\nu_1(E) \geq 0$.

$$\therefore \nu_1(E) = 0$$

— (vii)

From eqn (v) & (vii), we have

$$\mu(E) = 0 \quad \forall E \in \mathcal{B} \Rightarrow \nu_1(E) = 0$$

$\Rightarrow \nu_1$ is absolutely continuous w.r. to μ

$$\text{i.e. } \nu_1 \ll \mu.$$

For uniqueness of ν_0 & ν_1 :

Let ν'_0 and ν'_1 be measures such that

$$\nu = \nu'_0 + \nu'_1, \quad \nu'_0 \perp \mu \quad \text{and} \quad \nu'_1 \ll \mu$$

Then $\nu = \nu_0 + \nu_1$ and $\nu = \nu'_0 + \nu'_1$ are the two Lebesgue decomposition of ν .

So we have,

$$\nu_0 + \nu_1 = \nu'_0 + \nu'_1$$

$$\Rightarrow \nu_0 - \nu'_0 = \nu'_1 - \nu_1$$

Since ν_1 & ν'_1 are absolutely continuous w.r. to μ .

$\Rightarrow \nu'_1 - \nu_1$ is also absolutely continuous w.r. to μ .

$$\Rightarrow \mu(E) = 0 \Rightarrow (\nu'_1 - \nu_1)(E) = 0$$

$$\Rightarrow \nu'_1(E) - \nu_1(E) = 0$$

$$\Rightarrow \nu'_1(E) = \nu_1(E)$$

$$\Rightarrow \boxed{\nu_1 = \nu'_1}$$

Also, ν_0 and ν'_0 are singular w.r. to μ .

$\Rightarrow \nu_0 - \nu'_0$ is also singular w.r. to μ .

$$\Rightarrow \mu(E) = 0 = (\nu_0 - \nu'_0)(X-E)$$

$$\Rightarrow (\nu_0 - \nu'_0)(X-E) = 0$$

$$\Rightarrow \nu_0(X-E) - \nu'_0(X-E) = 0$$

$$\Rightarrow \nu_0(X-E) = \nu'_0(X-E)$$

$$\Rightarrow \boxed{\nu_0 = \nu'_0}$$

Hence ν_0 and ν_1 are unique.

Remark: • The decomposition $\nu = \nu_0 + \nu_1$ is called a Lebesgue decomposition of a σ -finite measure ν with respect to σ -finite measure μ .

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Theorem: Let A and B are two measurable sets and ν is a signed measure such that $A \subset B$ and $|\nu(B)| < \infty$. Then show that $|\nu(A)| < \infty$.

Proof:

Suppose A and B are two measurable sets and ν is a signed measure such that $A \subset B$. Then we can write,

$$B = (B-A) \cup A$$

$$\therefore \nu(B) = \nu(B-A) + \nu(A)$$

$$\Rightarrow |\nu(B)| = |\nu(B-A) + \nu(A)| \quad \text{--- (i)}$$

Given that $|\nu(B)| < \infty$

$$\Rightarrow |\nu(B-A) + \nu(A)| < \infty$$

[by (i)]

$$\Rightarrow |\nu(A)| < \infty$$

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* Characteristic Function :- Let E be any subset of X . Then characteristic function on E is defined as,

$$\chi_E(x) = \begin{cases} 1 & , \text{if } x \in E \\ 0 & , \text{if } x \in X-E \end{cases}$$

* Simple Function :- A linear combination

$$\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}$$

is called a simple function.

where the sets $E_i = \{x : \varphi(x) = c_i\}$ are disjoint and measurable and the number c_i are distinct and non-zero constants and χ_{E_i} is characteristic function of E_i ; $\forall i=1,2,\dots,n$.

* Lebesgue Bounded Convergence Theorem :- Let $\{f_n\}$ be a sequence

of measurable functions defined on a set E of finite measure. Suppose there is a real number M such that $|f_n(x)| \leq M$, $\forall x$ and $\forall n$.

If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ for each $x \in E$, then

$$\int_E f = \lim_{n \rightarrow \infty} \int_E f_n$$

* Lebesgue Monotone Convergence Theorem :- Let $\{f_n\}$ be a

increasing sequence of non-negative measurable functions and let $f = \lim_{n \rightarrow \infty} f_n$ a.e. on a set E . Then

$$\int f = \lim_{n \rightarrow \infty} \int f_n$$

* Riesz Representation Theorem :- Let F be a bounded linear functional on L^p ; $1 \leq p < \infty$. Then there is a function g in L^q such that

$$F(f) = \int fg$$

and also we have $\|F\| = \|g\|_q$.

OR

Let F be a bounded linear functional on $L^p(\mu)$ with $1 \leq p < \infty$ and μ be a σ -finite measure. Then there is a unique element g in $L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$F(f) = \int fg \, d\mu$$

and also we have $\|F\| = \|g\|_q$.

Proof :-

Let us first consider the case of finite measure μ .

Then every bounded measurable function is in $L^p(\mu)$.

Define a set function ν on the measurable set by setting,

$$\nu(E) = F(\chi_E) \quad \text{--- (i)}$$

where χ_E is the characteristic function on E .

Now, we have to show that ν is a signed measure.

(1.) Clearly, ν is an extended real valued function i.e. ν takes at most one value $-\infty$ and $+\infty$.

(20) In (i), take $E = \emptyset$, we have

$$\nu(\emptyset) = \int (X_\emptyset) = 0$$

$$\Rightarrow \nu(\emptyset) = 0.$$

(30) Let E be any measurable set and $\{E_n\}$ be a sequence of disjoint measurable sets such that $E = \bigcup_{n=1}^{\infty} E_n$.

Suppose $\alpha_n = \operatorname{sgn} F(X_{E_n})$. — (ii)

$$\text{where } \operatorname{sgn} c = \begin{cases} 0 & ; c = 0 \\ \frac{c}{|c|} & ; c \neq 0. \end{cases}$$

$$\text{Set } f = \sum_{n=1}^{\infty} \alpha_n \cdot X_{E_n}$$

$$\therefore F(f) = F\left(\sum_{n=1}^{\infty} \alpha_n \cdot X_{E_n}\right)$$

$$= \sum_{n=1}^{\infty} \alpha_n F(X_{E_n})$$

$$= \sum_{n=1}^{\infty} \operatorname{sgn} F(X_{E_n}) \cdot F(X_{E_n}) \quad [\text{by (ii)}]$$

$$= \sum_{n=1}^{\infty} \frac{F(X_{E_n})}{|F(X_{E_n})|} \cdot F(X_{E_n})$$

$$= \sum_{n=1}^{\infty} |F(X_{E_n})|$$

$$\Rightarrow F(f) = \sum_{n=1}^{\infty} \nu(E_n) \quad \text{— (iii)} \quad [\text{by (i)}]$$

$$\text{Now, } \|\alpha_n \cdot X_{E_n}\| = \|\alpha_n\| \cdot \|X_{E_n}\|$$

$$= 1 \cdot \|X_{E_n}\|$$

[by eqn (ii)]

$$= \left[\int_{E_n} |X_{E_n}|^p d\mu \right]^{1/p}$$

[$\because X_{E_n} \in L^p$]

$$= \left(\int_{E_n} d\mu \right)^{1/p}$$

[by defⁿ of characteristic]

$$\Rightarrow \|\alpha_n \cdot X_{E_n}\| = [\mu(E_n)]^{1/p}$$

$$\Rightarrow \|\alpha_n \cdot X_{E_n}\|^p = \mu(E_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} \|\alpha_n \cdot X_{E_n}\|^p = \sum_{n=1}^{\infty} \mu(E_n) = \mu(E)$$

$$\Rightarrow \sum_{n=1}^{\infty} \|\alpha_n \cdot X_{E_n}\|^p = \mu(E).$$

— (iv)

Since μ is finite
 i.e. $\mu(E) < \infty$
 $\Rightarrow \sum_{n=1}^{\infty} \|\alpha_n \cdot X_{E_n}\| < \infty$

Now by the result, "If $\{E_n\}$ be a sequence of disjoint measurable sets and for each n , let f_n be a function in L^p ; $1 \leq p < \infty$, that vanish outside of E_n , set $f = \sum_{n=1}^{\infty} f_n$. Then $f \in L^p$ iff $\sum_{n=1}^{\infty} \|f_n\|^p < \infty$ ".

Here we have $\sum_{n=1}^{\infty} \|\alpha_n \cdot X_{E_n}\| < \infty$
 and $(\alpha_n \cdot X_{E_n})$ is vanish outside of E_n .
 $\Rightarrow f \in L^p$.

Since F is bounded
 i.e. $\|F\| \leq M$ and $F(f) < \infty$ — (v)

From eqn (iii) & (v),
 $\sum_{n=1}^{\infty} |\nu(E_n)| < \infty$

By eqn (i), $\sum_{n=1}^{\infty} \nu(E_n) = \sum_{n=1}^{\infty} F(X_{E_n})$
 $= F(X_E)$
 $\Rightarrow \sum_{n=1}^{\infty} \nu(E_n) = \nu(E)$

i.e. ν satisfies the countable additivity property.

Thus ν is a signed measure.

If $\mu(E) = 0$

Then $\int X_E d\mu = \int_E d\mu = \mu(E) = 0$

$$\Rightarrow \int X_E d\mu = 0$$

$$\Rightarrow X_E = 0$$

$$\Rightarrow F(X_E) = 0$$

$$\Rightarrow v(E) = 0$$

[by (i)]

$$\text{i.e. } \mu(E) = 0 \Rightarrow v(E) = 0$$

$\Rightarrow v$ is absolutely continuous w.r. to μ .

By Radon Nikodym Theorem, $\exists$ a measurable function g such that for each measurable set E , we have

$$v(E) = \int_E g d\mu \quad \text{--- (vi)}$$

Since v is always finite.

$\Rightarrow g$ is integrable.

If ψ is a simple function,

then $\exists$ measurable sets $E_1, E_2, \dots, E_n$ such that

$$\psi = \sum_{i=1}^n a_i X_{E_i} \quad \text{--- (vii)}$$

$$\text{Now, } F(\psi) = \sum_{i=1}^n a_i F(X_{E_i})$$

$$= \sum_{i=1}^n a_i v(E_i)$$

[by (i)]

$$= \sum_{i=1}^n a_i \int_{E_i} g d\mu$$

[by (vi)]

$$= \sum_{i=1}^n a_i \int X_{E_i} g d\mu$$

$$= \int \left(\sum_{i=1}^n a_i X_{E_i} \right) g d\mu$$

$$= \int \psi g d\mu$$

[by (vii)]

i.e. If ψ is a simple function.

$$\text{Then } F(\psi) = \int \psi g d\mu.$$

$$\therefore \left| \int \psi g d\mu \right| = |F(\psi)| = \|F\| \cdot \|\psi\|_p \leq M \cdot \|\psi\|_p$$

[by (v)]

$$\Rightarrow \left| \int \psi g \, d\mu \right| \leq M \cdot \|\psi\|_p$$

for all simple function ψ .

Again by result, " If (X, β, μ) be a finite measure space and g is an integrable function such that for some constant M ,

$$\left| \int \psi g \, d\mu \right| \leq M \cdot \|\psi\|_p \quad \forall \psi$$

Then $g \in L^q$.

Therefore we get $g \in L^q$.

Let G be a bounded linear functional defined on L^p by

$$G(f) = \int f g \, d\mu.$$

Since F and G are bounded linear functional $\therefore G - F$ is also bounded linear functional, which vanish on the subspace of simple function.

$$[(G - F)\psi = G(\psi) - F(\psi) = \int \psi g \, d\mu - \int \psi g \, d\mu = 0]$$

Also, simple functions are dense in L^p .

$$\text{and } G - F = 0 \Rightarrow F = G.$$

Thus for each $f \in L^p$,

$$F(f) = \int f g \, d\mu.$$

and also we have

$$\|F\| = \|G\| = \|g\|_q$$

[By the result, " Let F be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$. Then $\exists$ unique element $g \in L^q$ such that $F(f) = \int f g \, d\mu$ for $f \in L^p$.

and we have $\|F\| = \|g\|_1$]

Thus g must be unique.

Now, we extend the theorem from finite measure to σ -finite measure.

For this, let $\{X_n\}$ be an increasing sequence of measurable sets of finite measure, whose union is X .

i.e. $X = \bigcup_{n=1}^{\infty} X_n$ such that $X_n \subseteq X_{n+1}$

The theorem for finite measure space implies that for each n , there is a function $g_n \in L^1$ such that g_n is vanish outside of X_n .

$$\text{and } F(f) = \int f g_n \, d\mu \quad \forall f \in L^p$$

$$\text{--- } \|F\| = \|g\|_1 \geq \|g_n\|_1$$

$$\Rightarrow \|F\| \geq \|g_n\|_1 \quad \text{--- (viii)}$$

Since any function g_n with all the properties is uniquely determined on X_n except for changes on sets of measure zero and g_{n+1} also has this property.

We may assume $g_{n+1} = g_n$ on X_n .

For $x \in X_n$,

$$\text{Set } g(x) = g_n(x)$$

$$\text{i.e. } \lim_{n \rightarrow \infty} g_n(x) = g(x).$$

Clearly, g is well define measurable function
and $|g_n|$ increases pointwise to $|g|$.

So by Lebesgue monotone convergence theorem,

$$\begin{aligned}\int |g|^q du &= \lim_{n \rightarrow \infty} \int |g_n|^q du \\ &= \lim_{n \rightarrow \infty} \left[\left(\int |g_n|^q du \right)^{1/q} \right]^q \\ &= \lim_{n \rightarrow \infty} \|g_n\|_q^q\end{aligned}$$

$$\Rightarrow \int |g|^q du \leq \|F\|^q \quad [\text{by (viii)}]$$

$$\Rightarrow \left(\int |g|^q du \right)^{1/q} \leq \|F\|$$

$$\Rightarrow \|g\|_q \leq \|F\| < \infty \quad \text{--- (ix)}$$

From eqn (viii) & (ix), we get

$$\boxed{\|F\| = \|g\|_q}$$

Let $f \in L^p$ and $f_n = f$ on X_n

i.e. $\lim_{n \rightarrow \infty} f_n = f$

And $f_n = 0$ on X_n^c

i.e. f_n is vanish outside of X_n .

Thus $f_n \rightarrow f$ pointwise in L^p .

Since $f \in L^p$ and $g \in L^q$

$\Rightarrow f$ and g are integrable

$\Rightarrow |fg|$ is integrable and also bounded
and $|f_n g| \leq |fg|$

By Lebesgue bounded convergence theorem,

$$\begin{aligned}\int fg du &= \lim_{n \rightarrow \infty} \int f_n g du \\ &= \lim_{n \rightarrow \infty} \int f_n g_n du \\ &= \lim_{n \rightarrow \infty} F(f_n)\end{aligned}$$

$$\Rightarrow \boxed{\int fg \, d\mu = F(f)} \quad \text{for } f \in L^p \text{ \& } g \in L^q$$

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Theorem: Let F be a bounded linear functional on $L^p(\mu)$ with $1 \leq p < \infty$. Then there is a unique element $g \in L^q$ such that

$$F(f) = \int fg \, d\mu$$

$$\text{We have } \|F\| = \|g\|_q$$

OR

D.T. Mann Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $g \in L^q(\mu)$, then the function F defined by

$$F(f) = \int fg \, d\mu \quad \text{for } f \in L^p(\mu)$$

is a bounded linear functional on $L^p(\mu)$ and

$$\|F\| = \|g\|_q$$

Proof:

First, we have to show that F is bounded linear functional on $L^p(\mu)$.

(1) Let $f_1, f_2 \in L^p(\mu)$.

$$\begin{aligned} \text{Then } F(f_1 + f_2) &= \int (f_1 + f_2)g \, d\mu \\ &= \int f_1 g \, d\mu + \int f_2 g \, d\mu \\ \Rightarrow F(f_1 + f_2) &= F(f_1) + F(f_2) \end{aligned}$$

$$(2) \quad F(af) = \int a \cdot fg \, d\mu = a \int fg \, d\mu$$

$$\Rightarrow F(af) = a \cdot F(f) \quad \text{where } a \text{ is scalar.}$$

Thus F is linear on $L^p(\mu)$.

Also we have

$$|F(f)| = \left| \int fg \, d\mu \right| \leq \|f\|_p \|g\|_q \quad [\text{By H\"older's Inequality}]$$

$$\Rightarrow \frac{|F(f)|}{\|f\|_p} \leq \|g\|_q \quad \text{s.t.} \quad \|f\|_p \neq 0$$

$$\Rightarrow \sup \left\{ \frac{|F(f)|}{\|f\|_p} : \|f\|_p \neq 0 ; f \in L^p \right\} \leq \|g\|_q$$

$$\Rightarrow \|F\| \leq \|g\|_q < \infty \quad \text{--- (i)} \quad [\because g \in L^q]$$

$$\Rightarrow \|F\| < \infty$$

Hence F is a bounded linear functional on $L^p(\mu)$.

Next, we have to show that $\|F\| \geq \|g\|_q$.

For this, let

$$f = |g|^{q-1} \operatorname{sgn} g \quad \text{--- (ii)}$$

$$\text{where } \operatorname{sgn} c = \begin{cases} 0 & ; \text{ if } c = 0 \\ \frac{c}{|c|} & ; \text{ if } c \neq 0. \end{cases}$$

$$\begin{aligned} \text{Then } |f|^p &= | |g|^{q-1} \operatorname{sgn} g |^p \\ &= | |g|^{q-1} |^p \\ &= |g|^{(q-1)p} \end{aligned}$$

$$\Rightarrow |f|^p = |g|^q \quad \text{--- (iii)} \quad \left[\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q = (q-1)p \right]$$

$$\Rightarrow \int |f|^p d\mu = \int |g|^q d\mu < \infty$$

$$\Rightarrow \int |f|^p d\mu < \infty$$

$$\Rightarrow f \in L^p(\mu).$$

$$\begin{aligned} \text{Also, } \|f\|_p &= \left[\int |f|^p d\mu \right]^{1/p} \\ &= \left[\int |g|^q d\mu \right]^{1/p} \quad \text{[By (iii)]} \\ &= \left[\left(\int |g|^q d\mu \right)^{1/q} \right]^{q/p} \\ &= \|g\|_q^{q/p} \end{aligned}$$

$$\Rightarrow \|f\|_p = \|g\|_q^{q-1} \quad \text{--- (iv)} \quad \left[\because \frac{1}{p} + \frac{1}{q} = 1 \right]$$

$$\begin{aligned} \text{Now } F(f) &= \int fg d\mu \\ &= \int (|g|^{q-1} \operatorname{sgn} g) g d\mu \quad \text{[by (ii)]} \end{aligned}$$

$$\Rightarrow F(f) = \int |g|^{p-1} \cdot \frac{g}{|g|} \cdot f \, d\mu$$

$$\begin{aligned} \Rightarrow |F(f)| &= \int |g|^p \, d\mu \\ &= \left[\left(\int |g|^p \, d\mu \right)^{1/p} \right]^p \\ &= \|g\|_q^p \\ &= \|g\|_q^{p-1} \cdot \|g\|_q \end{aligned}$$

$$\Rightarrow |F(f)| = \|f\|_p \cdot \|g\|_q \quad [\text{by (iv)}]$$

$$\Rightarrow \frac{|F(f)|}{\|f\|_p} = \|g\|_q \quad ; \quad \|f\|_p \neq 0 \quad \text{--- (v)}$$

By definition of $\|F\|$,

$$\|F\| = \sup \left\{ \frac{|F(f)|}{\|f\|_p} : \|f\|_p \neq 0 ; f \in L^p \right\}$$

$$\Rightarrow \|F\| \geq \frac{|F(f)|}{\|f\|_p} = \|g\|_q \quad [\text{by (v)}]$$

$$\Rightarrow \|F\| \geq \|g\|_q \quad \text{--- (vi)}$$

From eqn (i) & (vi), we have

$$\|F\| = \|g\|_q.$$

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Theorem: Let μ and ν are signed measure, then the following conditions are equivalent:

- (i) $\nu \ll \mu$
- (ii) $\nu^+ \ll \mu$ and $\nu^- \ll \mu$
- (iii) $|\nu| \ll |\mu|$

Proof:

For (i) $\Rightarrow$ (ii).

Suppose $\nu \ll \mu$

Then we have to show that $\nu^+ \ll \mu$ & $\nu^- \ll \mu$.

For this, let A be a measurable set such that

$$\mu(A) = 0.$$

Since $\nu \ll \mu$.

$$\therefore \mu(A) = 0 \quad \forall A \in \mathcal{A} \Rightarrow \nu(A) = 0.$$

$$\therefore \nu(A) = \nu^+(A) - \nu^-(A) = 0 \quad \text{--- (1)}$$

$$\therefore |\nu|(A) = \nu^+(A) + \nu^-(A) = 0. \quad \text{--- (2)}$$

By (1) and (2),

$$\nu^+(A) = \frac{1}{2} [\nu(A) + |\nu|(A)] = 0$$

$$\Rightarrow \nu^+(A) = 0$$

$$\text{and } \nu^-(A) = \frac{1}{2} [\nu(A) - |\nu|(A)] = 0$$

$$\Rightarrow \nu^-(A) = 0$$

$$\text{Thus } \mu(A) = 0 \Rightarrow \nu^+(A) = 0$$

$$\text{i.e. } \boxed{\nu^+ \ll \mu}$$

$$\text{and } \mu(A) = 0 \Rightarrow \nu^-(A) = 0$$

$$\text{i.e. } \boxed{\nu^- \ll \mu}$$

For (ii) $\Rightarrow$ (iii)

Suppose $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Then we have to show that $|\nu| \ll \mu$

For this, let A be a measurable set such that $\mu(A) = 0$

Then $\nu^+ \ll \mu$

$$\therefore \mu(A) = 0 \quad \forall A \in \mathcal{A} \Rightarrow \nu^+(A) = 0$$

and $\nu^- \ll \mu$

$$\therefore \mu(A) = 0 \quad \forall A \in \mathcal{A} \Rightarrow \nu^-(A) = 0$$

$$\text{Now } |\nu|(A) = \nu^+(A) + \nu^-(A) = 0$$

$$\Rightarrow |\nu|(A) = 0$$

$$\text{Thus } |\mu|(A) = 0 \Rightarrow |\nu|(A) = 0$$

$$\text{i.e. } \boxed{|\nu| \ll |\mu|}$$

For (iii) $\Rightarrow$ (i)

Suppose $|\nu| \ll |\mu|$

Then we have to show that $\nu \ll \mu$

For this, let A be a measurable set such that $\mu(A) = 0$

Since $|\nu| \ll |\mu|$

$$\therefore |\mu|(A) = 0 \quad \forall A \in \mathcal{A} \Rightarrow |\nu|(A) = 0$$

$$\Rightarrow \nu^+(A) + \nu^-(A) = 0$$

$$\Rightarrow \nu^+(A) = 0 \quad \text{and} \quad \nu^-(A) = 0$$

$$\text{Now } \nu(A) = \nu^+(A) - \nu^-(A) = 0$$

$$\Rightarrow \nu(A) = 0$$

$$\text{Thus } \mu(A) = 0 \quad \forall A \in \mathcal{A} \Rightarrow \nu(A) = 0$$

$$\text{i.e. } \boxed{\nu \ll \mu}$$

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* Semi Algebra :- A collection $\mathcal{G}$ of subsets of X is a semi algebra of sets if

(1.) The intersection of any two sets in $\mathcal{G}$ is again in $\mathcal{G}$.

$$\text{i.e. } A, B \in \mathcal{G} \Rightarrow A \cap B \in \mathcal{G}$$

(2.) The complement of any set in $\mathcal{G}$ is a finite disjoint union of sets in $\mathcal{G}$.

$$\text{i.e. } A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G} \text{ with } A^c = \bigcup_{i=1}^n A_i \in \mathcal{G}$$

Remarks:

- If $\mathcal{G}$ is any semi algebra of sets, then the collection α contains of the empty sets and all finite disjoint union of sets in $\mathcal{G}$ is an algebra of sets which is called algebra generated by $\mathcal{G}$.
- If μ is a set function define on semi algebra $\mathcal{G}$, it is natural to attempt to define a finitely additive set function on algebra α generated by $\mathcal{G}$ by setting

$$\mu(A) = \sum_{i=1}^n \mu(E_i)$$

where A is the disjoint union of sets E_i in $\mathcal{G}$
 i.e. $A = \bigcup_{i=1}^n E_i$.

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Proposition: Let $\mathcal{G}$ be a semi-algebra of sets and μ is a non-negative set function defined on $\mathcal{G}$ with $\mu(\emptyset) = 0$ [if $\emptyset \in \mathcal{G}$]. Then μ has a unique

extension to a measure on the algebra $\mathcal{G}$ generated by $\mathcal{C}$ if the following conditions are satisfied -

(1.) If a set C in $\mathcal{C}$ is the union of finite disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$.

Then
$$\mu(C) = \sum \mu(C_i).$$

(2.) If a set C in $\mathcal{C}$ is the union of countable disjoint collection $\{C_i\}$ of sets in $\mathcal{C}$.

Then
$$\mu(C) \leq \sum \mu(C_i).$$

* Local Measure :- If $(X, \mathcal{B}, \mu)$ is a measure space, then we say that a subset E of X is locally measurable if $E \cap B \in \mathcal{B}$ for each $B \in \mathcal{B}$ with $\mu(B) < \infty$.

Note :- • The collection $\mathcal{G}$ of all locally measurable sets is a semi algebra containing $\mathcal{B}$.

- The measure μ is called saturated if every locally measurable set is measurable.
- Every σ -finite measure is saturated.
- A measure can always be extended to a saturated measure.

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★. Baire Measure :- Let X be the set of real numbers and $\mathcal{B}$ be the class of all Borel sets. A measure μ defined on $\mathcal{B}$ and finite for bounded sets is called a Baire measure (on the real line).

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★. Cumulative Distribution Function :- Each finite Baire measure

associate a function F by setting

$$F(x) = \mu(-\infty, x]$$

The function F is called the cumulative distribution function of μ and is real-valued and monotone increasing. We have

$$\mu[a, b] = F(b) - F(a).$$

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★. Product Measure :- Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces and consider the direct product $X \times Y$ of X and Y .

- (1.) If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $A \times B$ is a rectangle.
- (2.) If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $A \times B$ is a measurable rectangle.
- (3.) The collection $\mathcal{R}$ of measurable rectangles is a semi algebra.
- (4.) If $A \times B$ is a measurable rectangle, then we set a function λ on $\mathcal{R}$, we have

$$\lambda(A \times B) = \mu(A) \cdot \nu(B)$$

(5.) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let $x \in X$, $y \in Y$ & $E \subset X \times Y$.

Then we write

$$E_x = \{y : \langle x, y \rangle \in E\}$$

$$\text{and } E_y = \{x : \langle x, y \rangle \in E\}$$

where E_x is called section of E determined by x or X -section.

and E_y is called section of E determined by y or Y -section.

- Both E_x and E_y are measurable sets.

(6.) Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces and let $x \in X$, $y \in Y$ & $E \subset X \times Y$ and let f be measurable function on E .

Then we write

$$f_x(y) = f(x, y)$$

$$\text{and } f_y(x) = f(x, y)$$

where $f_x(y)$ and $f_y(x)$ are functions over the sets E_x and E_y respectively.

And f_x is called X -section of measurable function of f & f_y is called Y -section of measurable function of f .

- Both f_x and f_y are measurable functions.

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* Definition :- Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two σ -finite measure spaces. Let $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $E \subset X \times Y$, then consider a real valued set function λ from $X \times Y$ to $\mathbb{R}$ i.e. $\lambda : X \times Y \rightarrow \mathbb{R}$ such that

$$\lambda(E) = \int \nu(E_x) d\mu = \int \mu(E_y) d\nu$$

and $\lambda(A \times B) = \mu(A) \cdot \nu(B).$

Then λ is a measure function and it is reflected to as product of μ and ν and also we write -

$$\lambda = \mu \times \nu.$$

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Lemma ⁽³⁰⁴⁾: Let $\{A_i \times B_i\}$ be a countable disjoint collection of measurable rectangles, whose union is a measurable rectangle $A \times B$. Then

$$\lambda(A \times B) = \sum_{i=1}^{\infty} \lambda(A_i \times B_i).$$

Proof:

Let (X, α, μ) and (Y, β, ν) be two complete measure spaces.

If $A \in \alpha$ and $B \in \beta$

For a fixed $x \in A$, for each $y \in B$, the point (x, y) belongs to exactly one rectangle $A_i \times B_i$.

Thus B is the disjoint union of B_i 's such that x is in the corresponding A_i .

i.e. $B = \bigcup_i B_i$

so $\nu(B) = \sum_i \nu(B_i)$

$\Rightarrow \nu(B) \chi_A(x) = \sum_i \nu(B_i) \chi_{A_i}(x).$ [By countable additivity prop.]

But we know that if $\{f_n\}$ be a sequence of non-negative measurable function.

Then by monotone convergence theorem,

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

$$\therefore \int v(B) \chi_A(x) du = \sum_{i=1}^{\infty} \int v(B_i) \chi_{A_i}(x) du$$

$$\Rightarrow v(B) \int \chi_A(x) du = \sum_{i=1}^{\infty} v(B_i) \int \chi_{A_i}(x) du$$

$$\Rightarrow v(B) \int_A du = \sum_{i=1}^{\infty} v(B_i) \int_{A_i} du$$

$$\Rightarrow v(B) \cdot \mu(A) = \sum_{i=1}^{\infty} v(B_i) \cdot \mu(A_i)$$

$$\Rightarrow \boxed{\lambda(A \times B) = \sum_{i=1}^{\infty} \lambda(A_i \times B_i)} \quad [\text{By definition}]$$

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Lemma: Let x be a point of X and E is a set in $\mathcal{R}_Y$. Then E_x is a measurable subset of Y .

Proof:

Let E is in the class of $\mathcal{R}$ of measurable rectangles. Then $\exists$ two measurable sets A and B in X and Y respectively such that

$$E = A \times B$$

i.e. E is a measurable rectangle.

Then $E_x = \{y : (x, y) \in E\}$.

• If $x \notin A$

Then $E_x = \emptyset$

• If $x \in A$

Then $E_x = B$.

Since $\emptyset$ and B are measurable subsets of Y .

$\therefore E_x$ is measurable subset of Y .

Next we show that if $E \in \mathcal{R}$,
then E_x is measurable subset of Y .

Let $E \in \mathcal{R}$
i.e. $E = \bigcup_{i=1}^{\infty} E_i$, where each E_i is measurable rectangle.

$$\begin{aligned}\text{Then } X_{E_x}(y) &= X_E(x, y) \\ &= \sup_i X_{E_i}(x, y) \\ \Rightarrow X_{E_x}(y) &= \sup_i X_{(E_i)_x}(y) \\ \text{i.e. } X_{E_x} &= \sup_i X_{(E_i)_x}\end{aligned}$$

Since each E_i is measurable rectangle.

$\therefore (E_i)_x$ is measurable subset of Y .

$\Rightarrow X_{(E_i)_x}$ is measurable for each i ,
and we know that supremum of measurable
functions be also measurable.

$\Rightarrow X_{E_x}$ is measurable.

$\Rightarrow E_x$ is measurable subset of Y .

Also we show that if $E \in \mathcal{R}_{\sigma}$
then E_x is measurable subset of Y .

Let $E \in \mathcal{R}_{\sigma}$

i.e. $E = \bigcap_{i=1}^{\infty} E_i$, where each $E_i \in \mathcal{R}$ i.e. $E_i = \bigcup_{j=1}^{\infty} E_{ij}$

$$\begin{aligned}\text{Then } X_{E_x}(y) &= X_E(x, y) \\ &= \inf_i X_{E_i}(x, y) \\ \Rightarrow X_{E_x}(y) &= \inf_i X_{(E_i)_x}(y) \\ \text{i.e. } X_{E_x} &= \inf_i X_{(E_i)_x}\end{aligned}$$

Since each E_i is measurable rectangle.

$\therefore (E_i)_x$ is measurable subset of Y .

$\Rightarrow \chi_{(E_i)_x}$ is measurable for each i ,
and we know that infimum of measurable functions be also measurable.

$\Rightarrow \chi_{E_x}$ is measurable.

$\Rightarrow E_x$ is measurable subset of Y .

Hence E_x is measurable subset of Y , for any $E \in \mathcal{R}_{\sigma}$.

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Lemma 1: Let E be a set in $\mathcal{R}_{\sigma}$ with $\mu \times \nu(E) < \infty$. Then the function g defined by $g(x) = \nu(E_x)$ is a measurable function of x and

$$\int g \, d\mu = \mu \times \nu(E)$$

Proof :

- Let E is a measurable rectangle
i.e. $E \in \mathcal{R}$.

Then $\exists$ two measurable sets A & B in X & Y respectively such that

$$E = A \times B$$

where A and B be measurable with respect to μ and ν respectively.

Now if $x \notin A$, then $E_x = \emptyset$

and if $x \in A$, then $E_x = B$

$$\text{So, } g(x) = \begin{cases} \nu(\emptyset) = 0 & , \text{ if } x \notin A \\ \nu(B) & , \text{ if } x \in A \end{cases} \quad \text{--- (i)}$$

Then $g(x) = v(B) \cdot \chi_A(x)$

$\Rightarrow g = v(B) \cdot \chi_A$

Since A is measurable set.

$\therefore \chi_A$ is a measurable function.

$\Rightarrow g$ is measurable function.

Now, $\int g \, du = \int v(B) \chi_A \, du = v(B) \int \chi_A \, du$

$= v(B) \int_A \, du = v(B) \cdot u(A)$

$\Rightarrow \int g \, du = u(A) \cdot v(B) = (u \times v)(A \times B)$

$\Rightarrow \int g \, du = u \times v(E)$

Thus the theorem is true for $E \in \mathcal{R}$.

• Now let $E \in \mathcal{R}_0$

i.e. $E = \bigcup_{i=1}^{\infty} E_i$, where each E_i is measurable rectangle.

Since every element of $\mathcal{R}_0$ is disjoint union of measurable rectangles.

Therefore $\exists$ a sequence $\{E_i\}$ of measurable rectangles such that $E = \bigcup_{i=1}^{\infty} E_i$

and set $g_i(x) = v((E_i)_x)$ — (ii)

Since each E_i is measurable rectangle.

$\therefore$ Each g_i is measurable function

and $\int g_i \, du = u \times v(E_i)$ — (iii)

Now $(\sum_i g_i)(x) = \sum_i g_i(x)$

$\Rightarrow (\sum_i g_i)(x) = \sum_i v((E_i)_x)$ [by (ii)] — (iv)

Since $E = \bigcup_i E_i$

$$\therefore E_x = \bigcup_i (E_i)_x$$

$$\Rightarrow \nu(E_x) = \sum_i \nu((E_i)_x) \quad \text{--- (v)}$$

From eqn (iv) & (v), we have

$$\left(\sum_i g_i\right)(x) = \nu(E_x)$$

$$\Rightarrow \left(\sum_i g_i\right)(x) = g(x)$$

$$\text{i.e.} \quad \sum_i g_i = g \quad \text{--- (vi)}$$

We know that sum of measurable function is also measurable.

So g is measurable.

By Monotone convergence theorem,

$$\int \sum_i g_i \, d\mu = \sum_i \int g_i \, d\mu$$

$$\Rightarrow \int g \, d\mu = \sum_i \mu \times \nu(E_i) \quad [\text{by (iii), (vi)}]$$

$$\Rightarrow \int g \, d\mu = \mu \times \nu(E)$$

Thus the theorem is true for $E \in \mathcal{R}_\sigma$.

• Now let $E \in \mathcal{R}_\sigma$

i.e. $E = \bigcup_i E_i$, where each $E_i \in \mathcal{R}_\sigma$ i.e. $E_i = \bigcup_j E_{ij}$

Then $\exists$ a sequence $\{E_i\}$ of sets in $\mathcal{R}_\sigma$ such that

$$E_{i+1} \subset E_i \quad (\text{Decreasing seq}^n)$$

$$\text{and} \quad \mu \times \nu(E_i) < \infty \quad \text{--- (vii)}$$

$$\text{Let } g_i(x) = \nu((E_i)_x) \quad \text{--- (viii)}$$

Then g_i is measurable function.

$$\therefore \int g_i \, d\mu = \mu \times \nu(E_i) < \infty \quad [\text{by (vii)}]$$

$$\text{i.e.} \quad \int g_i \, d\mu < \infty$$

So we have $\{ (E_i)_x \}$ is a decreasing sequence of measurable sets of finite measure, whose intersection is E_x .

$$\text{i.e. } \bigcap_{i=1}^{\infty} (E_i)_x = E_x$$

$$\text{Then } g(x) = \nu(E_x) = \nu(\bigcap_{i=1}^{\infty} (E_i)_x) = \lim_{i \rightarrow \infty} \nu((E_i)_x) \quad [\text{by (viii)}]$$

$$\Rightarrow g(x) = \lim_{i \rightarrow \infty} g_i(x)$$

$$\text{i.e. } g = \lim_{i \rightarrow \infty} g_i$$

$$\text{or } g_i \rightarrow g \text{ as } i \rightarrow \infty$$

$$\Rightarrow g \text{ is measurable.}$$

By Dominated convergence theorem, "let g be an integrable function on E and let $\{f_n\}$ be a seqⁿ of measurable function such that

$$|f_n| \leq g \text{ and } \lim_{n \rightarrow \infty} f_n = f \text{ a.e. on } E$$

$$\text{Then } \int_E f = \lim_{n \rightarrow \infty} \int_E f_n."$$

$$\text{So we have } \int g \, d\mu = \lim_{i \rightarrow \infty} \int g_i \, d\mu$$

$$= \lim_{i \rightarrow \infty} \mu \times \nu(E_i)$$

$$\Rightarrow \int g \, d\mu = \mu \times \nu(E)$$

Hence the theorem is true for $E \in \mathcal{R}_\sigma$.

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Pro. (I): Let μ be a measure on an algebra $\mathcal{A}$, μ^* the outer measure induced by μ and E any set. Then for $\varepsilon > 0$, there is a set $A \in \mathcal{A}$ with $E \subset A$ and

$$\mu^*(A) \leq \mu^*(E) + \varepsilon.$$

There is also a set $B \in \mathcal{A}$ with $E \subset B$
and $\mu^*(E) = \mu^*(B)$.

Proof :

Let μ be a measure on an algebra $\mathcal{A}$,
and μ^* be the outer measure induced by μ .
Also let E be any set.

Then by definition of μ^* , we have

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) : \{A_i\} \subset \mathcal{A} \text{ such that } E \subset \bigcup_{i=1}^{\infty} A_i \right\}$$

Since $E \subset \bigcup_{i=1}^{\infty} A_i$, so for $\varepsilon > 0$, by infimum lemma,
 $\sum_{i=1}^{\infty} \mu(A_i) < \mu^*(E) + \varepsilon$ — (i)

Set $A = \bigcup_{i=1}^{\infty} A_i$ i.e. $A \in \mathcal{A}$
 $\therefore \mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ [By σ -subadditivity]
 $\Rightarrow \mu^*(A) < \mu^*(E) + \varepsilon$ — (ii) [by (i)]
such that $E \subset A$.

Let $B = \bigcap_{n=1}^{\infty} A_n$

Then $B \in \mathcal{A}$ and $E \subset B$.

Since $B \subset A_n$

$$\therefore \mu^*(B) \leq \mu^*(A_n) < \mu^*(E) + \frac{1}{n}$$

$$\Rightarrow \mu^*(B) < \mu^*(E) + \frac{1}{n}$$

since n is arbitrary, so let $n \rightarrow \infty$, we get
 $\mu^*(B) \leq \mu^*(E)$ — (iii)

Also since $E \subset B$

$$\therefore \mu^*(E) \leq \mu^*(B)$$
 — (iv)

[By monotonicity prop.]

From eqn (iii) and (iv),
$$\mu^*(B) = \mu^*(E)$$

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(306)
Lemma 2.2: Let E be a set for which $\mu \times \nu(E) = 0$. Then
for almost all x , we have $\nu(E_x) = 0$.

Proof :

By proposition (T.), $\exists$ a set $F \in \mathcal{R}_0$ such that $E \subset F$
and $\mu \times \nu(E) = \mu \times \nu(F) = 0$.

By Lemma 1., we have

$$\begin{aligned} \int g \, d\mu &= \mu \times \nu(E) = \mu \times \nu(F) = 0 \\ \Rightarrow \int g \, d\mu &= 0 \quad \text{and} \quad g(x) = \nu(E_x) \\ \Rightarrow g &= 0 \text{ a.e.} \quad \text{and} \quad g(x) = \nu(E_x) \\ \Rightarrow g(x) &= \nu(E_x) = 0 \\ \Rightarrow \nu(E_x) &= 0 \text{ a.e.} \end{aligned}$$

OR

Let E be any measurable set.

Then $\exists$ measurable sets A & B in X & Y
respectively such that $E = A \times B$,

where A and B are measurable sets with respect
to μ and ν respectively.

Now if $x \notin A$, then $E_x = \emptyset$

and if $x \in A$, then $E_x = B$.

Now define a function $g(x) = \nu(E_x)$ such that

$$g(x) = \begin{cases} \nu(\emptyset) = 0 & , \text{ if } x \notin A \\ \nu(B) & , \text{ if } x \in A. \end{cases}$$

Then $g(x) = \nu(B) \chi_A(x)$

i.e. $g = \nu(B) \chi_A$

Now $\int g \, d\mu = \int \nu(B) \chi_A \, d\mu = \nu(B) \int \chi_A \, d\mu$
 $= \nu(B) \int \chi_A \, d\mu = \nu(B) \cdot \mu(A)$

$\Rightarrow \int g \, d\mu = \mu(A) \cdot \nu(B) = \mu_{X \times Y}(A \times B) = \mu_{X \times Y}(E) = 0$

$\Rightarrow \int g \, d\mu = 0$

$\Rightarrow g = 0 \quad \text{a.e.}$

$\Rightarrow g(x) = \nu(E_x) = 0 \quad \text{a.e.}$

$\Rightarrow \nu(E_x) = 0 \quad \text{a.e.}$

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Proposition (307): Let E be a measurable subset of $X \times Y$ such that $\mu_{X \times Y}(E)$ is finite. Then for almost all x the set E_x is a measurable subset of Y . The function g is defined by $g(x) = \nu(E_x)$ is a measurable function defined for almost all x and

$$\int g \, d\mu = \mu_{X \times Y}(E).$$

Proof:

Let E is in the class $\mathcal{R}$ of measurable rectangle. Then $\exists$ two measurable sets $A \subseteq X$ & $B \subseteq Y$ respectively.

Let $E = A \times B$

i.e. E is a measurable rectangle.

Then $E_x = \{y : (x, y) \in E\}$

Now if $x \notin A$, then $E_x = \emptyset$
 and if $x \in A$, then $E_x = B$

Since $\emptyset$ and B are measurable subset of Y .

Therefore E_x is measurable subset of Y . — (i)

Define a function $g(x) = \nu(E_x)$ such that

$$g(x) = \begin{cases} \nu(\emptyset) = 0 & , \text{ if } x \notin A \\ \nu(B) & , \text{ if } x \in A \end{cases}$$

Then $g(x) = \nu(B) \cdot \chi_A(x)$

i.e. $g = \nu(B) \cdot \chi_A$

Since A is measurable set,

$\therefore \chi_A$ is measurable function.

$\Rightarrow g$ is also measurable function.

$$\begin{aligned} \text{Now } \int g \, d\mu &= \int \nu(B) \cdot \chi_A \, d\mu = \nu(B) \int \chi_A \, d\mu \\ &= \nu(B) \int_A 1 \, d\mu = \nu(B) \cdot \mu(A) \end{aligned}$$

$$\Rightarrow \int g \, d\mu = \mu(A) \cdot \nu(B) = \mu \times \nu (A \times B)$$

$$\Rightarrow \int g \, d\mu = \mu \times \nu (E) \quad \text{--- (ii)}$$

This completes the proof.

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*. Fubini Theorem :- Let (X, α, μ) and (Y, β, ν) be two complete measure spaces

and f be an integrable function on $X \times Y$. Then

(1.) For almost all x the function f_x defined by

$$f_x(y) = f(x, y)$$

is an integrable function on Y .

(2.) For almost all y the function f^y defined by

$$f_y \text{ or } f^y = f(x, y)$$

is an integrable function on X .

- (3.) $\int_Y f(x, y) \cdot d\nu(y)$ is an integrable function on X .
 (4.) $\int_X f(x, y) \cdot d\mu(x)$ is an integrable function on Y .
 (5.) $\int_X [\int_Y f d\nu] d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y [\int_X f d\mu] d\nu$

Proof:

Because of the symmetry between x and y , it is sufficient to prove 1st, 3rd and the first half of 5th.

- (1.) First suppose that f is a characteristic function on measurable set of finite measure.

Let X_E be such function -

$$f = X_E \text{ and } \mu \times \nu(E) < \infty$$

$$\begin{aligned} (1.) \text{ Now } f_x(y) &= f(x, y) = X_E(x, y) = X_{E_x}(y) \\ \Rightarrow f_x(y) &= X_{E_x}(y) \\ \text{i.e. } f_x &= X_{E_x} \quad \text{--- (i)} \end{aligned}$$

Since E is measurable set.

$\therefore E_x$ is measurable subset of Y .

$\Rightarrow X_{E_x}$ is measurable function on Y .

$\Rightarrow f_x$ is measurable on Y . [by (i)]

$$\begin{aligned} \text{Also } \int_Y f_x(y) d\nu(y) &= \int_Y X_{E_x}(y) d\nu(y) \quad \text{[by (i)]} \\ &= \int_{E_x} d\nu(y) \\ \Rightarrow \int_Y f_x(y) d\nu(y) &= \nu(E_x) \quad \text{--- (ii)} \end{aligned}$$

Since E_x is measurable subset of Y .

and E has finite measure i.e. $\mu \times \nu(E) < \infty$

$\Rightarrow E_x$ also has finite measure i.e. $\nu(E_x) < \infty$ --- (iii)

From eqn (ii) and (iii),

$$\int_Y f_x(y) d\nu(y) = \int_Y f(x,y) d\nu(y) = \nu(E_x) < \infty \quad \text{--- (iv)}$$

i.e. f_x is integrable on Y .

^x(2.) Next suppose that f is a characteristic function on measurable set of finite measure.

Let X_E be such function

$$f = X_E \quad \text{and} \quad \mu\nu(E) < \infty$$

$$\text{Now } f^y(x) = f(x,y) = X_E(x,y) = X_{E_y}(x)$$

$$\Rightarrow f^y(x) = X_{E_y}(x)$$

$$\text{i.e. } f^y = X_{E_y} \quad \text{--- (v)}$$

Since E is measurable set.

$\therefore E_y$ is measurable subset of X .

$\Rightarrow X_{E_y}$ is measurable function on X

$\Rightarrow f^y$ is measurable on X . [by (v)]

$$\text{Also } \int_X f^y(x) d\mu(x) = \int_X X_{E_y}(x) d\mu(x) \quad \text{[by (v)]}$$
$$= \int_{E_y} d\mu(x)$$

$$\Rightarrow \int_X f^y(x) d\mu(x) = \mu(E_y) \quad \text{--- (vi)}$$

Since E_y is measurable subset of X

and E has finite measure i.e. $\mu\nu(E) < \infty$

$\Rightarrow E_y$ also has finite measure i.e. $\mu(E_y) < \infty$ --- (vii)

From eqn (vi) and (vii),

$$\int_X f^y(x) d\mu(x) = \int_X f(x,y) d\mu(x) = \mu(E_y) < \infty \quad \text{--- (viii)}$$

i.e. f^y is integrable on X .

(3.) We know that a function g is defined by

$$g(x) = v(E_x) \quad \text{a.e.} \quad \text{--- (ix)}$$

and g is measurable.

$$\therefore \int g \, d\mu = \mu_{Xv}(E)$$

$$\Rightarrow \int_X g(x) \, d\mu(x) = \mu_{Xv}(E)$$

$$\Rightarrow \int_X v(E_x) \, d\mu(x) = \mu_{Xv}(E) \quad [\text{by (ix)}]$$

$$\Rightarrow \int_X \left[\int_Y f_x(y) \, dv(y) \right] d\mu(x) = \mu_{Xv}(E) < \infty \quad [\text{by (iv)}]$$

$\Rightarrow \int_Y f_x(y) \, dv(y)$ is integrable over X .

* (4.) We know that a function g is defined by

$$g(y) = \mu(E_y) \quad \text{a.e.} \quad \text{--- (x)}$$

and g is measurable.

$$\therefore \int g \, dv = \mu_{Xv}(E)$$

$$\Rightarrow \int_Y g(y) \, dv(y) = \mu_{Xv}(E)$$

$$\Rightarrow \int_Y \mu(E_y) \, dv(y) = \mu_{Xv}(E) \quad [\text{by (x)}]$$

$$\Rightarrow \int_Y \left[\int_X f^y(x) \, d\mu(x) \right] dv(y) = \mu_{Xv}(E) < \infty \quad [\text{by (viii)}]$$

$\Rightarrow \int_X f^y(x) \, d\mu(x)$ is integrable over Y .

$$(5.) \text{ Now } \int_X \left[\int_Y f_x(y) \, dv(y) \right] d\mu(x) = \mu_{Xv}(E)$$

$$= \int_E d(\mu_{Xv})$$

$$\Rightarrow \int_X \left[\int_Y f_x(y) \, dv(y) \right] d\mu(x) = \int_{X \times Y} \chi_E(x, y) \, d(\mu_{Xv}) \quad \text{--- (xi)}$$

$$* \text{ [Also, } \int_Y \left[\int_X f^y(x) \, d\mu(x) \right] dv(y) = \mu_{Xv}(E)$$

$$= \int_E d(\mu_{Xv})$$

$$\Rightarrow \int_Y \left[\int_X f^y(x) \, d\mu(x) \right] dv(y) = \int_{X \times Y} \chi_E(x, y) \, d(\mu_{Xv})$$

$$\Rightarrow \int_Y \left[\int_X f^y(x) \, d\mu(x) \right] dv(y) = \int_{X \times Y} f(x, y) \, d(\mu_{Xv}) \quad \text{--- (xii)}$$

From eqn (xi) and (xii),

$$\int_x \left[\int_y f_x(y) \, d\nu(y) \right] d\mu(x) = \int_{xy} f(x,y) \, d(\mu \times \nu) = \int_y \left[\int_x f^y(x) d\mu(x) \right] d\nu(y)$$

Thus the theorem is true for characteristic function.

Since we know that, "Every simple function can be express as a linear combination of a characteristic function".

So this theorem is true for all simple function.

Also we know that each non-negative integrable function f is the limit of an increasing seqⁿ of non-negative simple function.

So $\exists$ non-negative sequence of simple function $\{\phi_n\}$ such that

$$\phi_n \rightarrow f \quad \text{as } n \rightarrow \infty$$

Since each ϕ_n is integrable and simple function. So it must be vanish outside a set of finite measure.

[$\because$ Every simple function vanish outside a set of finite measure.]

Thus f_x is the limit of the increasing sequence $(\phi_n)_x$.

By Monotone convergence theorem,

$$\int_y f_x(y) \, d\nu(y) = \int_y f(x,y) \, d\nu(y)$$

$$\bullet \frac{d}{dy} \left(\frac{y}{x^2+y^2} \right) = \frac{x^2+y^2 - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\bullet \frac{d}{dx} \left(\frac{-x}{x^2+y^2} \right) = \frac{-x^2-y^2 + x \cdot 2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \int_Y f_x(y) \, d\nu(y) = \lim_{n \rightarrow \infty} \int_Y (\phi_n)_x(y) \, d\nu(y)$$

$$\Rightarrow \int_Y f_x(y) \, d\nu(y) = \lim_{n \rightarrow \infty} \int_Y \phi_n(x, y) \, d\nu(y).$$

Since $\int_Y (\phi_n)_x(y) \, d\nu(y)$ is integrable function, and limit of integrable funⁿ is also integrable.

$\Rightarrow \int_Y f_x(y) \, d\nu(y)$ is integrable on X . — (xiii)

Similarly, we have

$\int_X f^y(x) \, d\mu(x)$ is integrable on Y . — (xiv)

Again by Monotone convergence theorem,

$$\int_X \left[\int_Y f \, d\nu \right] d\mu = \lim_{n \rightarrow \infty} \int_X \left[\int_Y \phi_n \, d\nu \right] d\mu$$

$$= \lim_{n \rightarrow \infty} \int_{X \times Y} \phi_n \, d(\mu \times \nu)$$

$$\Rightarrow \int_X \left[\int_Y f \, d\nu \right] d\mu = \int_{X \times Y} f \, d(\mu \times \nu)$$

$$\text{Similarly } \int_Y \left[\int_X f \, d\mu \right] d\nu = \int_{X \times Y} f \, d(\mu \times \nu)$$

$$\text{Thus } \int_X \left[\int_Y f \, d\nu \right] d\mu = \int_Y \left[\int_X f \, d\mu \right] d\nu = \int_{X \times Y} f \, d(\mu \times \nu)$$

Que. : Explain why -

$$\int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dx \right] dy \neq \int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dy \right] dx$$

Soln :

$$\text{Let } f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\begin{aligned} \text{Now } \int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dy &= \int_0^1 \frac{d}{dy} \left(\frac{y}{x^2+y^2} \right) dy \\ &= \left[\frac{y}{x^2+y^2} \right]_0^1 \\ &= \frac{1}{1+x^2} \end{aligned}$$

$$\therefore \int_0^1 \left[\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2} \, dy \right] dx = \int_0^1 \frac{1}{1+x^2} \, dx = \left[\tan^{-1} x \right]_0^1$$

$$\Rightarrow \int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right] dx = \frac{\pi}{4} \quad \text{--- (i)}$$

$$\begin{aligned} \text{And } \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx &= \int_0^1 \frac{d}{dx} \left(\frac{-x}{x^2 + y^2} \right) dx \\ &= \left[\frac{-x}{x^2 + y^2} \right]_0^1 \\ &= \frac{-1}{1 + y^2} \end{aligned}$$

$$\begin{aligned} \therefore \int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \right] dy &= \int_0^1 \frac{-1}{1 + y^2} dy \\ &= - \left[\tan^{-1} y \right]_0^1 \end{aligned}$$

$$\Rightarrow \int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \right] dy = -\frac{\pi}{4} \quad \text{--- (ii)}$$

From eqn (i) and (ii),

$$\int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx \right] dy \neq \int_0^1 \left[\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dy \right] dx$$

The reason for this that f is not Lebesgue integrable or integrable over $[0, 1] \times [0, 1]$.

* Lebesgue Stieltjes Integral :

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Lemma: ²⁰⁰ If μ is a finite Baire measure on the real line, then its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right.

Moreover, $\lim_{x \rightarrow -\infty} F(x) = 0$.

Proof:

The cumulative distribution function F is given by $F(x) = \mu(-\infty, x]$ — (i)

If $x \leq y$,

then $(-\infty, x] \subseteq (-\infty, y]$

$\therefore \mu(-\infty, x] \leq \mu(-\infty, y]$

$\Rightarrow F(x) \leq F(y)$

Thus F is monotonic increasing function.

Now, for $(a, b]$ i.e. $a \leq b$

$$\begin{aligned} F(b) - F(a) &= \mu(-\infty, b] - \mu(-\infty, a] \\ &= \mu[(-\infty, b] - (-\infty, a]) \\ &= \mu[(-\infty, b] \cap (-\infty, a]^c] \end{aligned}$$

$$\Rightarrow F(b) - F(a) = \mu(a, b] \quad \text{--- (ii)}$$

Now, since $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}]$

$$\therefore \mu(a, b] = \mu\left[\bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}]\right]$$

$$\Rightarrow \mu(a, b] = \lim_{n \rightarrow \infty} \mu(a, b + \frac{1}{n}] \quad \text{--- (iii)}$$

$$\text{By (ii) \& (iii), } F(b) - F(a) = \lim_{n \rightarrow \infty} \mu(a, b + \frac{1}{n}]$$

$$\Rightarrow F(b) - F(a) = \lim_{n \rightarrow \infty} [F(b + \frac{1}{n}) - F(a)]$$

$$\Rightarrow F(b) - F(a) = \lim_{n \rightarrow \infty} F(b + \frac{1}{n}) - F(a)$$

$$\Rightarrow F(b) = \lim_{n \rightarrow \infty} F(b + \frac{1}{n}) = F(b^+)$$

Thus F is continuous on the right.

$$\begin{aligned}
 \text{Similarly, } \mu\{b\} &= \lim_{n \rightarrow \infty} \mu(b - \frac{1}{n}, b] \\
 &= \lim_{n \rightarrow \infty} [F(b) - F(b - \frac{1}{n})] \\
 &= F(b) - \lim_{n \rightarrow \infty} F(b - \frac{1}{n}) \\
 &\Rightarrow \mu\{b\} = F(b) - F(b^-).
 \end{aligned}$$

Thus F is continuous at b iff the set $\{b\}$ consisting of F be along has measure zero.

$$\begin{aligned}
 \text{Now, } \phi &= \bigcap_{n=1}^{\infty} (-\infty, -n] \\
 \therefore \mu(\phi) &= \lim_{n \rightarrow \infty} \mu(-\infty, -n] \\
 \Rightarrow 0 &= \lim_{n \rightarrow \infty} F(-n) \quad [\text{by (i)}] \\
 \Rightarrow \lim_{n \rightarrow -\infty} F(n) &= 0 \\
 \text{i.e. } \lim_{x \rightarrow -\infty} F(x) &= 0
 \end{aligned}$$

Hence F is monotone increasing bounded function that is continuous on the right.

Lemma: Let F be a monotone increasing function continuous on the right. If $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i]$, then

$$F(b) - F(a) \leq \sum_{i=1}^{\infty} [F(b_i) - F(a_i)]$$

Proof:

The cumulative distribution function F is given by $F(x) = \mu(-\infty, x]$ — (i)

If $x \leq y$, then $F(x) \leq F(y)$.

Now, for $(a, b]$ i.e. $a \leq b$.

$$\begin{aligned}
 F(b) - F(a) &= \mu(-\infty, b] - \mu(-\infty, a] \quad [\text{by (i)}] \\
 &= \mu[(-\infty, b] \cap (-\infty, a]^c] \\
 \Rightarrow F(b) - F(a) &= \mu(a, b] \quad \text{--- (ii)}
 \end{aligned}$$

Since $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i]$

$$\therefore \mu(a, b] \leq \sum_{i=1}^{\infty} \mu(a_i, b_i] \quad [\text{By countable subadditivity}]$$

$$\Rightarrow F(b) - F(a) \leq \sum_{i=1}^{\infty} [F(b_i) - F(a_i)] \quad [\text{by eq (ii)}]$$

Proposition: Let F be a monotone increasing function which is continuous on the right. Then there is a unique Baire measure μ such that for all a and b we have $\mu(a, b] = F(b) - F(a)$.

Proof:

Here we see that μ admits a unique extension to a measure of the algebra generated by $\mathcal{G}$, then μ can be extended to a σ -algebra containing semi-algebra $\mathcal{G}$.

Let $\mathcal{G}$ be the set of all intervals of the form $(a, b]$ and (a, ∞) .

Then $\mathcal{G}$ is semi-algebra.

i.e. (i) $(a, b] \cap (c, d] \in \mathcal{G}$

$$(ii) (a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{G}$$

Let μ be a measure on $\mathcal{G}$ defined by

$$\mu(a, b] = F(b) - F(a).$$

also if $(a, b] \subset \bigcup_{i=1}^{\infty} (a_i, b_i]$

$$\text{then } F(b) - F(a) \leq \sum_{i=1}^{\infty} [F(b_i) - F(a_i)]$$

$$\Rightarrow \mu(a, b] \leq \sum_{i=1}^{\infty} \mu(a_i, b_i]$$

$$\text{Take } (a, b] = (a, b_i] \cup (b_i, b]$$

$$\therefore \mu(a, b] = \mu(a, b_i] + \mu(b_i, b]$$

$$\begin{array}{c} \frac{1}{b} + \frac{1}{b} \\ \hline \frac{1}{b} + \frac{1}{b} \end{array}$$

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So by proposition, μ admits a unique extension to a measure on algebra.

Also, by Carathéodory extension theorem, μ can be extended to a measure on a σ -algebra containing $\mathcal{B}$.

i.e. we have an extension μ to a Baire measure.

Also, μ is a σ -finite set because the set X is the union of $(n, n+1]$.

$$\text{i.e. } X = \bigcup_n (n, n+1]$$

$$\therefore \mu(n, n+1] = F(n+1) - F(n) < \infty$$

Hence extension of μ is unique.

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Theorem: If a function f defined on $[a, b]$ is absolutely continuous in $[a, b]$ and $f'(x) = 0$ almost everywhere, then f is constant.

Proof:

Let f be an absolutely continuous function in $[a, b]$ and $f'(x) = 0$ a.e. on $[a, b]$.

Then we have to show that f is constant.

$$\text{i.e. } f(c) = f(a) \text{ for any } c \in [a, b].$$

For this, let $E = \{x \in [a, c], f'(x) = 0\}$

Then $E \subset [a, c] \subset [a, b]$.

Now for any $x \in E$, $f'(x) = 0$

$$\Rightarrow \exists n, h > 0 \text{ such that } \frac{|f(x+h) - f(x)|}{h} < n$$

$$\Rightarrow |f(x+h) - f(x)| < \eta h$$

So corresponding to each $x \in E$, $\exists$ a small closed interval $[x, x+h]$ contained in $[a, b]$ such that

$$|f(x+h) - f(x)| < \eta h \quad \text{--- (i)}$$

$\Rightarrow$ The interval $[x, x+h]$ covers E .

$\therefore$ We can find a finite collection of pairwise disjoint closed interval $I_1 = [x_1, y_1]$, $I_2 = [x_2, y_2]$, ..., $I_n = [x_n, y_n]$, which covers all of E except for a set of measure less than δ , where $\delta > 0$.

If we consider $x_i \leq x_{i+1}$

Then we have

$$a = y_0 \leq x_1 < y_1 \leq x_2 < y_2 \leq \dots \leq x_n < y_n \leq x_{n+1} = c$$

and

$$\sum_{i=0}^n |x_{i+1} - y_i| < \delta \quad \text{for} \quad \sum_{i=0}^n |f(x_{i+1}) - f(y_i)| < \epsilon$$

$$\text{Now, } \sum_{i=0}^n |f(y_i) - f(x_i)| < \eta \cdot \sum_{i=0}^n (y_i - x_i) \quad [\text{by (i)}]$$

$$\Rightarrow \sum_{i=0}^n |f(y_i) - f(x_i)| < \eta (c - a)$$

$$\Rightarrow \sum_{i=0}^n |f(y_i) - f(x_i)| < \eta (b - a) \quad [\because c \in [a, b]]$$

$$\begin{aligned} \therefore |f(c) - f(a)| &= \left| \sum_{i=0}^n [f(x_{i+1}) - f(y_i)] + \sum_{i=0}^n [f(y_i) - f(x_i)] \right| \\ &\leq \sum_{i=0}^n |f(x_{i+1}) - f(y_i)| + \sum_{i=0}^n |f(y_i) - f(x_i)| \\ &< \epsilon + \eta (b - a) \end{aligned}$$

Since ϵ & η are arbitrary small, so $\epsilon, \eta \rightarrow 0$,

$$|f(c) - f(a)| = 0$$

$$\Rightarrow f(c) = f(a)$$

$\Rightarrow f$ is constant.

19-20/2.5

Que.: If ν is a finite signed measure and μ is a signed measure such that $\nu \ll \mu$ on E . Then there is a positive number δ such that $\nu(E) < \infty$ for every measurable set E for which $\mu(E) < \delta$.

Proof:

Since $\nu \ll \mu$

$$\therefore \mu(E) = 0 \quad \forall \quad E \in \mathcal{A} \Rightarrow \nu(E) = 0 \quad \text{--- (i)}$$

{ Let E be a measurable set with $\mu(E) < \delta$
Then we prove that $\nu(E) < \infty$

From eqn (i),

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

$$\text{and } \mu(E) < \delta \Rightarrow \nu(E) = 0 < \infty$$

$$\Rightarrow \mu(E) < \delta \Rightarrow \nu(E) < \infty$$

Hence, for any positive number δ ,
if $\mu(E) < \delta$ then $\nu(E) < \infty$.

21-22/1.5

* Totally Finite Signed Measure :- A signed measure μ is said to be a totally finite signed measure if a set $E \subset X$ is measurable and $|\mu(E)| < \infty$.

* Complex Measure :- A complex measure on the class of all measurable sets of a measurable space is a set function μ such that for every measurable set such that

$$\mu(E) = \mu_1(E) + i\mu_2(E)$$

where μ_1 and μ_2 are signed measure and $i = \sqrt{-1}$.

20-21/2-5

Theorem: If μ and ν are totally finite measure such that $\nu \ll \mu$ and ν is not identically zero, then $\exists$ a positive number $\varepsilon > 0$, and a mble set A such that $\mu(A) > 0$ and such that A is positive set for a signed measure $(\nu - \varepsilon\mu)$.

Proof:

Let μ and ν are totally finite measure such that $\nu \ll \mu$ and ν is not identically zero.

Then we have to show that $\exists$ a measurable set A such that $\mu(A) > 0$.

Since ν be a measure and $\nu \ll \mu$

Let (X, β, μ) be a σ -finite measure space and $E \subset X$.

So by Radon-Nikodym Theorem,

$\exists$ a non-negative measurable function f such that

$$\nu(E) = \int_E f \cdot d\mu \quad \forall E \in \beta \quad \text{--- (i)}$$

Since ν is not identically zero,

$\Rightarrow \exists$ a measurable set B such that

$$\nu(B) = \int_B f \cdot d\mu > 0$$

$$\text{Let } \varepsilon = \frac{1}{2} \inf_{x \in B} f(x) \quad \text{--- (ii)}$$

Now define the set

$$A = \{x \in B : f(x) \geq 2\varepsilon\}$$

where $\varepsilon > 0$ is a positive constant to be chosen later.

Claim : $\mu(A) > 0$

Verification :

Suppose on contrary, $\mu(A) = 0$

Then for any $E \subset A$, we have

$$\nu(E) = \int_E f d\mu \geq \int_E 2\varepsilon d\mu = 2\varepsilon \mu(E)$$

If $E = B$

$$\therefore \nu(B) \geq 2\varepsilon \mu(B)$$

which contradicts eqn (i). $\varepsilon < \frac{1}{2} \inf f(x) \leq \frac{1}{2} f(x)$

$$\therefore \mu(A) > 0$$

Now let $x \notin A$

Then $x \notin B$ or $f(x) < 2\varepsilon$

~~Doubt~~

Here we case,

$$(\nu - \varepsilon\mu)(A) = \int_A (\nu - \varepsilon) d\mu = (\nu - \varepsilon)\mu$$

Thus we have a positive number $\varepsilon > 0$

and a set A s.t. $\mu(A) > 0$

and A is positive for the signed measure $(\nu - \varepsilon\mu)$.

$\bar{O}$ = Complement of O .

[19/10/23] [21/10/23]

Bhavika Gupta

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* Locally Compact Space :- A topological space X is said to be locally compact space if for each $x \in X$, there is an open set O containing x such that $\bar{O}$ is compact.

Note :-

- Set X is locally compact iff the collection of open sets with compact closure forms a base for the topology X .
- Every compact space is locally compact, but locally compact space is not necessarily compact.

Exa. :- Euclidean space $\mathbb{R}^n$ is locally compact space but not compact space.

- A subset S of a topological space X is compact if for every open cover of S there exists a finite sub-cover of S .

* Hausdorff Space :- A topological space X is said to be Hausdorff if for every two distinct points x and y of X , there exists two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

* $C_c(X)$:- Let X be a locally compact Hausdorff space. Then the family $C_c(X)$ consisting of all continuous real valued functions that vanish outside a compact subset of X . Thus $C_c(X)$ is the class of all continuous real-valued functions defined on X with compact support.

* Support of f or Support :- If f is a real valued funⁿ. Then the support of f is the closure of the set $\{x : f(x) \neq 0\}$.

* The class of Baire sets $B_a(X)$:- The class of Baire sets is defined to be the smallest σ -algebra β of subsets of X such that each function in $C_c(X)$ is measurable with respect to β .

Thus β is the smallest σ -algebra generated by the sets $\{x : f(x) \geq \alpha\}$ with $f \in C_c(X)$ and $\alpha \in \mathbb{R}$.

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33) * Baire Sets :- Let β is the smallest σ -algebra generated by the sets $\{x : f(x) \geq \alpha\}$ with $f \in C_c(X)$ and $\alpha \in \mathbb{R}$. Then the members of β are called Baire sets.

Note :- If $\alpha > 0$

Then the sets $\{x : f(x) \geq \alpha\}$ are compact G_δ 's.

where G_δ = Countable intersection of open sets.

- Each compact G_δ set is a Baire set.
- $\mathcal{B}$ is the σ -algebra generated by the compact G_δ 's.

* The class of Borel sets $\mathcal{B}_0(X)$:- Let X is any topological space. Then the smallest σ -algebra containing the closed sets is said to be the class of Borel sets.

Borel Sets :- Let $\mathcal{B}$ be the smallest σ -algebra containing the closed sets. Then the members of $\mathcal{B}$ are called Borel sets.

Note :

- If X is locally compact, then every Baire set is Borel set.

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- By a Baire measure on X , we mean a measure defined for all Baire sets and finite for each compact Baire set.
- By a Borel measure on X , we mean a measure defined on the σ -algebra of Borel sets.

* Bounded Set :- A set E in a locally compact Hausdorff space is said to be a bounded set if E is contained in some compact set i.e. E is compact.

* σ -Bounded Set :- A set E is said to be σ -bounded if it is the union of countable collection of bounded sets.

* σ -Compact Set :- A set E is said to

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be a σ -compact set if the set E is the countable union of compact sets.

Note: • A set E is σ -bounded if and only if it is contained in a σ -compact set.

Lemma ³³²: Let K be a compact set, O an open set with $K \subset O$. Then

$$K \subset U \subset H \subset O$$

where U is a σ -compact open set and H is a compact G_δ .

Proof:

Let K be a compact set and O be an open set with $K \subset O$.

Then $\exists$ a bounded open set V such that $K \subset V \subset O$.

Without loss of generality, we assume that O is bounded.

Now, let f be any real-valued function such that

$$f(x) = 0 \quad \forall x \in K$$

$$f(x) = 1 \quad \forall x \in O^c$$

$$\text{Let } U = \{x : f(x) < \frac{1}{2}\}$$

$$\text{and } H = \{x : f(x) \leq \frac{1}{2}\}$$

$$\text{Clearly, } K \cup U \subset H \subset O.$$

$$\text{Since } H = \{x : f(x) \leq \frac{1}{2}\}$$

It follows that H is closed set. — (i)

If f is continuous.

$$\text{Then } H_n = \{x : f(x) \leq \frac{1}{2} + \frac{1}{n}\}$$

$$\therefore H = \bigcap H_n$$

$\Rightarrow H$ is G δ set. — (ii)

Also, since O is bounded and $H \subset O$.

$\Rightarrow H$ is bounded — (iii)

From eqn (i) and (iii), we get

H is closed and bounded

$\Rightarrow H$ is compact. — (iv)

Thus by eqⁿ (iii) & (iv),
H is compact G_δ set.

Next, we take $U_n = \{x : f(x) < \frac{1}{2} - \frac{1}{2^n}\}$
∴ $U = \bigcup_{n=1}^{\infty} U_n$

Since we know that,
H is compact set and $U \subset H$
⇒ $U = \bigcup_{n=1}^{\infty} U_n$ is compact set for each n.

So U is expressed as countable union of compact sets.

⇒ U is σ-compact set.

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Remark:

- Every σ-compact open set is the union of a countable collection of compact G_δ's and hence a Baire set.
- Every bounded set is contained in a compact G_δ. Every σ-bounded set E is contained in a σ-compact open

set O . If E is bounded, then $\bar{O}$ is compact.

- If E is a Baire set, then E or E^c is σ -bounded. Both are σ -bounded if and only if X is σ -compact.
- The class of σ -bounded Baire sets is the smallest σ -ring containing the compact G's.
- Each σ -bounded Baire set is the union of a countable disjoint union of bounded Baire sets.

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The Regularity of Baire & Borel Measure :

Let μ be a measure defined on a σ -algebra $\mathcal{M}$ of subsets of X and suppose that $\mathcal{M}$ contains the Baire sets.

[16/8*] [19/1.5+] [14/8-]

- Outer Regular : A set $E \in \mathcal{M}$ is said to be an outer regular

for μ [or μ is outer regular for E]
if

$$\mu(E) = \mu E = \inf \{ \mu O : E \subset O ; O \text{ open ; } O \in \mathcal{M} \}$$

[16/8*] [19/1.5] [14/8-]

• Inner Regular : A set $E \in \mathcal{M}$ is said to be an inner regular for μ [or μ is inner regular for E]
if

$$\mu(E) = \mu E = \sup \{ \mu K : K \subset E ; K \text{ compact ; } K \in \mathcal{M} \}$$

[17/1.5] [21/1.5]

• Regular : A set $E \in \mathcal{M}$ is said to be regular for μ if E is both inner and outer regular for μ .

[17/2.5] [18-19/2.5] [22-23/2.5] [14/8] [15/8]

Proposition: Let μ be a finite measure defined on a σ -algebra $\mathcal{M}$ which contains all the Baire sets of a locally compact space X . If μ is inner regular, it is regular.
[inner regular $\Rightarrow$ outer regular]

Proof: Let $E \in \mathcal{M}$

Since μ is inner regular,
Then by definition of inner regular,
 $\mu(E) = \sup \{ \mu(K) : K \subset E ; K \text{-compact} ; K \in \mathcal{M} \}$

But for such a K , we have $\bar{K}$ is open
and $E \subset \bar{K}$.

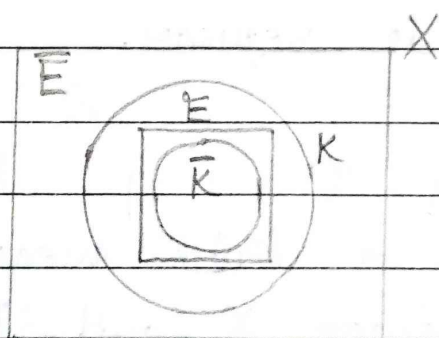


fig. - 01.

$$\begin{aligned} \text{So } \mu(E) &= \mu(X) - \mu(E^c) \\ &= \inf \{ \mu(X) - \mu(E^c) \} \\ &= \inf \{ \mu(X) - \mu(K^c) \} \\ &= \inf \{ \mu(X - K) \} \end{aligned}$$

$$\Rightarrow \mu(E) = \inf \{ \mu(\bar{K}) \}$$

$$\Rightarrow \mu(E) \geq \inf \{ \mu(O) : E \subset O ; O \text{-open} ; O \in \mathcal{M} \}$$

— (i)

Now if $E \subset O$.

Then by monotonicity property,

$$\mu(E) \leq \mu(O)$$

$$\Rightarrow \mu(E) \leq \inf \{ \mu(O) : E \subset O ; O \text{-open} ; O \in \mathcal{M} \} \quad \text{--- (ii)}$$

From eqn (i) & (ii),

$$\mu(E) = \inf \{ \mu(O) : E \subset O ; O \text{-open} ; O \in \mathcal{M} \}$$

$\Rightarrow \mu$ is outer regular.

Hence μ is regular.

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Theorem:

Let μ be a Baire measure on a locally compact space X and E a σ -bounded Baire set in X . Then for $\varepsilon > 0$

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(1.) There is a σ -compact open set O with $E \subset O$ and $\mu(O - E) < \varepsilon$

(2.) $\mu(E) = \sup \{ \mu(K) : K \subset E ; K \text{ is compact Gs} \}$

Proof:

Let $\mathcal{K}$ be the class of sets E that satisfies (1.) and (2.) for each $\varepsilon > 0$.

Then we shall show that it is also true for a set, which is countable union of

members of $\mathcal{R}$.

Suppose $E = \bigcup E_n$
where $E_n \in \mathcal{R}$.

Then by (1.), for each n , there is
 σ -compact open set O_n with $E_n \subset O_n$
and $\mu(O_n - E_n) < \varepsilon \cdot 2^{-n}$

Taking $O = \bigcup O_n$

Then O is also σ -compact open set with $E \subset O$
and $O - E \subset \bigcup (O_n - E_n)$

$$\therefore \mu(O - E) \leq \sum \mu(O_n - E_n)$$

$$< \sum 2^{-n} \cdot \varepsilon$$

$$= \varepsilon$$

$$[\because \sum 2^{-n} = 1]$$

$$\Rightarrow \mu(O - E) < \varepsilon$$

Thus E satisfies condition (1.).

Now,

- If for some n , we have $\mu(E_n) = \infty$

Then there are compact G's of arbitrary
large finite measure contained in $E_n \subset E$.

Thus condition (2.) holds for E .

• If $\mu(E_n) < \infty$

So for each n , there is a $K_n \subset E_n$,
 K_n is a compact Gs set
and

$$\mu(E_n) = \sup \{ \mu(K_n) \}$$

$$\Rightarrow \mu(E_n) - 2^{-n} \cdot \varepsilon < \mu(K_n) \quad [\text{by supremum lemma}]$$

$$\Rightarrow \mu(E_n) - \mu(K_n) < 2^{-n} \cdot \varepsilon$$

$$\Rightarrow \mu(E_n - K_n) < 2^{-n} \cdot \varepsilon \quad \text{--- (i)}$$

$$\text{Now, } \mu(E) = \sup_N \mu\left(\bigcup_{n=1}^N E_n\right)$$

$$\Rightarrow \mu(E) < \sup_N \mu\left(\bigcup_{n=1}^N K_n\right) + \varepsilon \quad [\text{by (i)}]$$

$\therefore$ It is true for every ε .

$$\Rightarrow \mu(E) \leq \sup_N \mu\left(\bigcup_{n=1}^N K_n\right)$$

$$\leq \sup \{ \mu(K) : K \subset E ; K \text{ - compact Gs, } K \in \mathcal{M} \}$$

--- (ii)

Taking $K = \bigcup K_n$

Then K is compact Gs with $K_n \subset E_n$

and $K \subset E$

$$\therefore \mu(K) \leq \mu(E)$$

$$\Rightarrow \sup \{ \mu(K) : K \subset E ; K \text{ - compact Gs ; } K \in \mathcal{M} \} \leq \mu(E)$$

--- (iii)

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From eqn (ii) and (iii), we have

$$\mu(E) = \sup \{ \mu(K) : K \subseteq E; K \text{ - compact Gs. } K \in \mathcal{M} \}$$

Thus E satisfies condition (2.1).

Now, if E is a compact Gs.

Then there a continuous real valued function ϕ with compact support $[C_c(X)]$ such that

$$0 \leq \phi \leq 1$$

$$\text{and } E = \{x : \phi(x) = 1\}$$

$$\text{Let } O_n = \{x : \phi(x) > 1 - \frac{1}{n}\}$$

Then,

It is clear that O_n is the inverse image of $(1 - \frac{1}{n}, \infty)$ w.r. to ϕ .

Also, O_n is a σ -compact open set with $\bar{O}_n$ compact.

$\because O_n$ is inverse image of $(1 - \frac{1}{n}, \infty)$ or $(-\infty, 1 - \frac{1}{n})$ w.r. to ϕ and ϕ is continuous

We know that continuous image of open set is open.

$\Rightarrow \phi^{-1}(-\infty, 1 - \frac{1}{n}]$ is open.

$\Rightarrow O_n$ is open.

Since $E = \bigcap O_n$

$$\therefore \mu(E) = \inf \mu(O_n) = \lim_{n \rightarrow \infty} \mu(O_n)$$

So for each $\varepsilon > 0$, $\exists$ a σ -compact open set O_n such that

$$\mu(O_n) - \varepsilon < \mu(E)$$

$$\Rightarrow \mu(O_n) - \mu(E) < \varepsilon$$

$$\Rightarrow \mu(O_n - E) < \varepsilon$$

Thus each compact G_δ satisfies condition (1.)
and trivially satisfies condition (2.)
i.e. $\mu(E) = \sup \{ \mu(K) : K \subset E; K \text{ compact } G_\delta \}$

Since E itself a compact G_δ set.

So every compact G_δ set satisfies condition (1.) and (2.).

Thus $E \in \mathcal{R}$.

Let X be a compact.

Claim: E satisfies (1.) iff $\bar{E}$ satisfies (2.).

Verification:

Suppose E satisfies (1.).

If possible suppose $\bar{E}$ does not satisfy (2.)

Then $\exists \varepsilon > 0$ such that

$$\mu(K) < \mu(\bar{E}) - \varepsilon \quad [\text{by (2.) \&supremum lemma}]$$

$$\forall \text{ compact } G \text{ s.t. } K \subset \bar{E}$$

$$\Rightarrow \varepsilon < \mu(\bar{E}) - \mu(K)$$

$$\Rightarrow \varepsilon < \mu(\bar{E} - K)$$

$$\Rightarrow \varepsilon < \mu(E \cap K)$$

$$\Rightarrow \varepsilon < \mu(K \cap \bar{E})$$

$$\Rightarrow \varepsilon < \mu(K - E) \quad \forall \text{ compact } G \text{ s.t. } K$$

Here $\bar{E}$ does not satisfy (1.).

which is contradiction of our assumption.

Thus if X is compact

Then E satisfies (1.) iff $\bar{E}$ satisfies (2.).

$\Rightarrow \mathcal{K}$ is a σ -algebra containing the compact G_δ sets.

Since all Baire sets are generated by compact G_δ -sets.

Therefore, $\mathcal{K}$ contains all Baire sets and theorem is hold, when X is compact.

By Lemma 1.

Next, for an arbitrary locally compact space X and bounded Baire set E , we can take H to be a compact G_δ set and U be a σ -bounded open subset of X such that

$$\overline{E} \subset U \subset H$$

$$[\Rightarrow E \subset U \subset H]$$

Then E is a Baire subset of H .

So $\exists$ a σ -compact open subset W such that $\mu(W-E) < \varepsilon$

Suppose $O = W \cap U$

Then O is a σ -compact subset with $E \subset O \subset W$

$$\begin{aligned} \circ \quad O-E &\subset W-E \\ \text{and } \mu(O-E) &\leq \mu(W-E) < \varepsilon \\ \Rightarrow \mu(O-E) &< \varepsilon \end{aligned}$$

Thus E satisfies (1-) and trivially (2-).

$\circ$ All bounded Baire sets are in $\mathcal{K}$.

But $\mathcal{K}$ is a σ -bounded set

$\circ$ E is also σ -bounded set

$\Rightarrow E$ is also in $\mathcal{K}$.

This completes the proof.

Remark:

- Every Baire measure is regular measure.
- If X is σ -compact, the σ -ring and the σ -algebra generated by the compact G's are the same.
- If X is σ -compact, then every Baire measure on X is regular.

Ex:

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★ Quasi Regular :- A measure μ defined on a σ -algebra $\mathcal{M}$ which contains the Baire sets is said to be quasi regular if it is outer regular and each open set $O \in \mathcal{M}$ is inner regular for μ .

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Proposition: Let μ be a measure defined on a σ -algebra $\mathcal{M}$ containing the Baire sets. Assume either that μ is quasi regular or that μ is inner regular. Then for each $E \in \mathcal{M}$ with $\mu(E) < \infty$, there is a Baire set B with

$$\mu(E \Delta B) = 0$$

Proof:

Case I. If μ is a quasi regular.

Let E be a measurable set of finite measure.

Since μ is outer regular,
so we can find a sequence $\{O_n\}$ of

open sets with

$$E \subset O_{n+1} \subset O_n$$

Now, by definition of outer regular,

$$\mu(E) = \inf \{ \mu(O_n) \}$$

$\therefore$ by infimum lemma,

$$\mu(O_n) < \mu(E) + 2^{-n}$$

—(i)

Also, since μ is quasi regular.

$\Rightarrow$ It is inner regular for all open sets.

$\therefore$ there is a compact set $K_m \subset O_m$ such that

$$\mu(O_m) = \sup \{ \mu(K_m) \}$$

So by supremum lemma,

$$\mu(K_m) > \mu(O_m) - 2^{-m}$$

Let we take K_m be a compact G δ -set.

i.e. $K_m = \cap O_m$

$$\therefore \mu(K_m) > \mu(O_m) - 2^{-m}$$

$$\geq \mu(E) - 2^{-m}$$

$$\Rightarrow \mu(K_m) > \mu(O_n) - 2^{-n} - 2^{-m} \quad \text{--- (ii) [by (i)]}$$

$$\text{Set } H_m = \bigcup_{j=m}^{\infty} K_j$$

Since each K_j is compact G δ set and we know that every compact G δ -set is a Baire set.

$\Rightarrow H_m$ is a Baire set such that

$$H_m \subset O_m \subset O_n, \quad m \geq n$$

$$[\because H_m = \bigcup_{j=m}^{\infty} K_j ; K_m \subset O_m]$$

Also, $H_m \supset H_{m+1}$ and $K_m \subset H_m$

$$\text{i.e. } H_m \supset K_m$$

$$\therefore \mu(H_m) \geq \mu(K_m) > \mu(O_n) - 2^{-n} - 2^{-m} \quad \text{--- (iii) [by (ii)]}$$

$$\text{Let } B = \bigcap H_m$$

Since each H_m is Baire set, so B is also Baire set with $B \subset O_n$.

$$\therefore \mu(B) = \lim_{m \rightarrow \infty} \mu(H_m)$$

$$\Rightarrow \mu(B) > \lim_{m \rightarrow \infty} (\mu(O_n) - 2^{-n} - 2^{-m}) \quad \text{[by (iii)]}$$

$$\Rightarrow \mu(B) \geq \mu(O_n) - 2^{-n}$$

$$\Rightarrow \mu(O_n) - \mu(B) \leq 2^{-n}$$

Since $B \subset O_n$ and $E \subset O_n$, we have
 $B \Delta E \subset (O_n \sim B) \cup (O_n \sim E)$

$$\therefore \mu(B \Delta E) \leq \mu(O_n \sim B) + \mu(O_n \sim E) \\ < 2^{-n} + 2^{-n}$$

This is true for any n , so $n \rightarrow \infty$

$$\boxed{\mu(B \Delta E) = 0}$$

Case II. If μ is an inner regular.

Then for $\varepsilon = \frac{1}{n}$, $\exists$ a compact set $K_n \in \mathcal{M}$
 such that $K_n \subset E$

$$\text{and } \mu(E) = \sup \{ \mu(K_n) \}$$

By supremum lemma,

$$\mu(K_n) > \mu(E) - \frac{1}{n}$$

$$\Rightarrow \mu(E) - \mu(K_n) < \frac{1}{n}$$

$$\Rightarrow \mu(E - K_n) < \frac{1}{n}$$

— (iv)

Since we know that for each compact set K and open set $O \supset K$, $\exists$ a σ -compact open set U and compact G_δ -set H such that

$$K \subset U \subset H \subset O$$

From this property, we have a σ -compact open set K_n' such that

$$K_n \subset K_n' \subset E \quad \text{--- (v)}$$

$$\text{Let } B = \bigcup K_n'$$

and we know that, every σ -compact Baire set is also a Baire set.

$\Rightarrow B$ is a Baire set.

$$\therefore \mu(E-B) \leq \mu(E-K_n')$$

$$\leq \mu(E-K_n)$$

[by (v)]

$$< \frac{1}{n}$$

[by (iv)]

Also, we have $B \subset E$

$$\Rightarrow \mu(B-E) \sim \mu(E-B)$$

$$\text{So, } \mu(B \Delta E) = \mu(B-E) + \mu(E-B)$$

$$\Rightarrow \mu(B \Delta E) < \frac{1}{n} + \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow \mu(B \Delta E) = 0 \quad \text{--- (vi)}$$

$$\text{But always } \mu(B \Delta E) \geq 0 \quad \text{--- (vii)}$$

From eqn (vi) and (vii), we have

$$\boxed{\mu(B \Delta E) = 0}$$

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Note : Let $\bar{\mu}$ be a non-negative extended real valued function defined on the class of open subsets of X and satisfying

$$(i) \quad \bar{\mu}(O) < \infty \text{ if } O \text{ compact.}$$

$$(ii) \quad \bar{\mu}(O_1) < \bar{\mu}(O_2) \text{ if } O_1 \subset O_2$$

$$(iii) \quad \bar{\mu}(O_1 \cup O_2) = \bar{\mu}(O_1) + \bar{\mu}(O_2) \text{ if } O_1 \cap O_2 = \emptyset$$

$$(iv) \quad \bar{\mu}(\cup O_i) \leq \sum \bar{\mu}(O_i).$$

$$(v) \quad \bar{\mu}(O) = \sup \{ \bar{\mu}(U) : U \subset O, U \text{ compact} \}$$

Then the set function μ^* defined by

$$\mu^*(E) = \inf \{ \bar{\mu}(O) : E \subset O \}$$

is a topologically regular outer measure.

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★. Borel Set :- Borel set is any set in topological space which can be found from open set or closed set through the operation of countable union, countable intersection and relative complement.

[21/9.5] [20-21/1.5]

Theorem : Every Borel set is σ -bounded and every σ -bounded open set is a Borel set.

Proof :

Part I. Since we know that,

Every Borel set is compact,
and a set which is contained in compact set is called bounded set,
and one which is contained in σ -compact is called σ -bounded.

Therefore every Borel set is σ -bounded.

Part II. Suppose O is an open set and let $\{C_n\}$ be a sequence of compact sets such that

$$O \subset \bigcup_{n=1}^{\infty} C_n = K \text{ (say)}$$

$$\text{Then } \bigcup_{n=1}^{\infty} C_n - O \in K$$

$$\text{Let } D = \bigcup_{n=1}^{\infty} C_n - O$$

Since O and $\bigcup_{n=1}^{\infty} C_n$ be compact.

$\Rightarrow D$ is also compact set.

$$\text{Since } D = K - O$$

$$\Rightarrow D = K \cap O^c \quad \Rightarrow D \text{ is open set}$$

$$\Rightarrow O = K - D$$

$$\text{i.e. } O = K \cap D^c$$

Clearly, O is countable intersection of open set and closed set.

So by definition of Borel set,

$\Rightarrow O$ is a Borel set.

Thus every σ -bounded open set is a Borel set.

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Theorem: Every finite disjoint union of inner regular sets of finite measure is inner regular.

Proof:

Let $\{E_i\}$ be a finite disjoint class of inner regular sets of finite measure.

Then by definition of inner regular,

$\exists$ a sequence of compact sets $\{C_i\}$ such that $C_i \subset E_i$; $\forall i \in \mathbb{N}$

$$\mu(E_i) = \sup \{ \mu(C_i) : C_i \subset E_i ; C_i \text{ is compact} \} \quad \forall i \in \mathbb{N}$$

So by supremum lemma,

$$\mu(C_i) > \mu(E_i) - \varepsilon/n$$

$$\Rightarrow \mu(E_i) < \mu(C_i) + \varepsilon/n \quad \text{--- (i)}$$

$$\text{Let } C = \bigcup_{i=1}^n C_i$$

$$\text{and } E = \bigcup_{i=1}^n E_i$$

$$\text{Now, } C_i \subset E_i$$

$$\Rightarrow \bigcup_{i=1}^n C_i \subset \bigcup_{i=1}^n E_i$$

$$\Rightarrow C \subset E$$

Since $\{C_i\}$ be a seqⁿ of compact sets

$\Rightarrow \bigcup_{i=1}^{\infty} C_i = C$ is also compact.

Since $E = \bigcup_{i=1}^{\infty} E_i$

So by countable subadditive property of μ ,

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

$$< \sum_{i=1}^n [\mu(C_i) + \epsilon/n] \quad [\text{by } C_i]$$

$$= \mu(C) + \sum_{i=1}^n \epsilon/n$$

$$= \mu(C) + \epsilon$$

$$\Rightarrow \mu(E) < \mu(C) + \epsilon$$

$\therefore$ By supremum lemma,

$$\mu(E) = \sup \{ \mu(C) : C \subset E ; C \text{ is compact} \}$$

Clearly, by defⁿ of inner regular,

$\Rightarrow E$ is inner regular.

i.e. $\bigcup_{i=1}^{\infty} E_i$ is inner regular.

20/H

Theorem^o The intersection of sequence of inner regular sets of finite measure is inner regular.

Also, the intersection of a decreasing seqⁿ of outer regular sets of finite measure is outer regular.

Proof:

Part I. Let $\{E_i\}$ be a finite disjoint class of inner regular sets of finite measure.

Then by definition of inner regular,
 $\exists$ a sequence $\{C_i\}$ of compact sets
such that $C_i \subset E_i \quad \forall i \in \mathbb{N}$
and $\mu(E_i) = \sup \{ \mu(C_i) : C_i \subset E_i ;$
 $C_i \text{ is compact } \forall i \in \mathbb{N} \}$

So by supremum lemma,

$$\begin{aligned} \mu(C_i) &> \mu(E_i) - \epsilon_i \\ \Rightarrow \mu(E_i) &< \mu(C_i) + \epsilon_i \end{aligned} \quad \text{--- (i)}$$

$$\begin{aligned} \text{Let } C &= \bigcup_{i=1}^{\infty} C_i \\ \text{and } E &= \bigcup_{i=1}^{\infty} E_i \end{aligned}$$

$$\begin{aligned} \text{Now, } C_i &\subset E_i \\ \Rightarrow \bigcup_{i=1}^{\infty} C_i &\subset \bigcup_{i=1}^{\infty} E_i \\ \Rightarrow C &\subset E \end{aligned}$$

Since $\{C_i\}$ be a seqⁿ of compact sets.
 $\Rightarrow \bigcup_{i=1}^{\infty} C_i = C$ is also compact.

Since $E = \bigcap_{i=1}^{\infty} E_i$

$$\therefore \mu(E) = \lim_{i \rightarrow \infty} \mu(E_i)$$

$$< \lim_{i \rightarrow \infty} [\mu(C_i) + \varepsilon] \quad [\text{by (i)}]$$

$$\Rightarrow \mu(E) < \mu(C) + \varepsilon$$

$\therefore$ By supremum lemma,

$$\mu(E) = \sup \{ \mu(C) : C \subset E ; C \text{ is compact} \}$$

Clearly, by definition of inner regular,
 $\Rightarrow E$ is inner regular.

i.e. $\bigcap_{i=1}^{\infty} E_i$ is inner regular.

Part II. Let $\{E_i\}$ be a decreasing sequence of outer regular of finite measure.

Then by definition of outer regular,

$\exists$ a sequence $\{O_i\}$ of open sets such that $E_i \subset O_i \quad \forall i \in \mathbb{N}$

and $\mu(E_i) = \inf \{ \mu(O_i) : E_i \subset O_i ; O_i \text{ is open} \}$

So by infimum lemma,

$$\mu(O_i) < \mu(E_i) + \varepsilon$$

$$\Rightarrow \mu(E_i) > \mu(O_i) - \varepsilon \quad \text{--- (ii)}$$

Let $E = \bigcap_{i=1}^{\infty} E_i$ and $O = \bigcap_{i=1}^{\infty} O_i$

$$\begin{aligned} \text{Now, } E_i &\subset O_i \\ \Rightarrow \bigcap_{i=1}^{\infty} E_i &\subset \bigcap_{i=1}^{\infty} O_i \\ \Rightarrow E &\subset O \end{aligned}$$

Since $\{O_i\}$ be a seqⁿ of open sets.
 $\Rightarrow \bigcap_{i=1}^{\infty} O_i = O$ is also open.

$$\begin{aligned} \text{Since } E &= \bigcap_{i=1}^{\infty} E_i \\ \therefore \mu(E) &= \lim_{i \rightarrow \infty} \mu(E_i) \\ &> \lim_{i \rightarrow \infty} [\mu(O_i) - \varepsilon] \\ &= \mu(O) - \varepsilon \\ \Rightarrow \mu(E) &> \mu(O) - \varepsilon \end{aligned}$$

$\therefore$ By infimum lemma,
 $\mu(E) = \inf \{ \mu(O) : E \subset O ; O \text{ is open} \}$

Clearly, by definition of outer regular,
 $\Rightarrow E$ is outer regular.
 i.e. $\bigcap_{i=1}^{\infty} E_i$ is outer regular.

Theorem: Prove that the intersection of increasing sequence of inner regular set is inner regular and the union of sequence of outer regular set is outer regular.

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22-24	2.5

Proof :

Part I. Let $\{E_i\}$ be a increasing sequence of inner regular sets.

Then by definition of inner regular,
 $\exists$ a sequence $\{C_i\}$ of compact sets such
that $C_i \subset E_i \quad \forall i \in \mathbb{N}$

and $\mu(E_i) = \sup \{ \mu(C_i) : C_i \subset E_i ; C_i \text{ is compact} \}$

So by supremum lemma,

$$\mu(C_i) > \mu(E_i) - \varepsilon_i$$

$$\Rightarrow \mu(E_i) < \mu(C_i) + \varepsilon_i \quad \text{--- (i)}$$

$$\text{Let } C = \bigcup_{i=1}^{\infty} C_i$$

$$\text{and } E = \bigcup_{i=1}^{\infty} E_i$$

$$\text{Now } C_i \subset E_i$$

$$\Rightarrow \bigcup_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} E_i$$

$$\Rightarrow C \subset E$$

Since $\{C_i\}$ be a sequence of compact sets.

$\Rightarrow \bigcup_{i=1}^{\infty} C_i = C$ is also compact.

$$\text{Since } E = \bigcup_{i=1}^{\infty} E_i$$

$$\therefore \mu(E) = \lim_{i \rightarrow \infty} \mu(E_i)$$

$$\Rightarrow \mu(E) < \lim_{i \rightarrow \infty} [\mu(C_i) + \varepsilon_i] \quad [\text{by (i)}]$$

$$= \mu(C) + 0$$

$$\Rightarrow \mu(E) < \mu(C)$$

$\therefore$ By supremum lemma,

$$\mu(E) = \sup \{ \mu(C) : C \subset E ; C \text{ is compact} \}$$

Clearly, by definition of inner regular,
 $\Rightarrow E$ is inner regular.

i.e. $\bigcup_{i=1}^{\infty} E_i$ is also inner regular.

Part II. Let $\{E_i\}$ be a ~~union~~ of sequence of outer regular.

Then by definition of outer measure,

$\exists$ a sequence $\{O_i\}$ of open sets such that $E_i \subset O_i \quad \forall i \in \mathbb{N}$

and $\mu(E_i) = \inf \{ \mu(O_i) : E_i \subset O_i ; O_i \text{ is open} \}$

So by infimum lemma,

$$\mu(O_i) < \mu(E_i) + \varepsilon$$

$$\Rightarrow \mu(E_i) > \mu(O_i) - \varepsilon$$

— (ii)

$$\text{Let } F = \bigcup_{i=1}^{\infty} E_i$$

$$\text{and } O = \bigcup_{i=1}^{\infty} O_i$$

$$\begin{aligned}
 \text{Now} \quad & E_i \subset O_i \\
 \Rightarrow \quad & \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} O_i \\
 \Rightarrow \quad & E \subset O
 \end{aligned}$$

Since $\{O_i\}$ be a sequence of open sets.
 $\Rightarrow \bigcup_{i=1}^{\infty} O_i = O$ is also open.

$$\begin{aligned}
 \text{Since} \quad & E = \bigcup_{i=1}^{\infty} E_i \\
 \therefore \quad & \mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \\
 & > \sum_{i=1}^{\infty} [\mu(O_i) - \varepsilon] \quad [\text{by (ii)}] \\
 & = \mu(O) - \varepsilon \\
 \Rightarrow \quad & \mu(E) > \mu(O) - \varepsilon
 \end{aligned}$$

$\therefore$ By infimum lemma,
 $\mu(E) = \inf \{ \mu(O) : E \subset O ; O \text{ is open} \}$

Clearly, by definition of outer regular,
 $\Rightarrow E$ is outer regular.
 i.e. $\bigcup_{i=1}^{\infty} E_i = E$ is also outer regular.

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* Regular (Topologically) :

An outer measure μ^* on a locally compact Hausdorff space X is said to be a regular (topologically) if

- (i) For each $E \subset X$,

$$\mu^*(E) = \inf \{ \mu^*(O) \mid O \text{ open, } E \subset O \}$$
- (ii) $\mu^*(O_1 \cup O_2) = \mu^*(O_1) + \mu^*(O_2)$
 if O_1 and O_2 are disjoint open sets.
- (iii) $\mu^*(O) = \sup \{ \mu^*(K) \mid K \subset O, K \text{ compact} \}$
 for O - open.

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Proposition: Let μ be a non-negative extended real valued function define on the class of open subsets of X and satisfying

- (i) $\mu(O) < \infty$, if $\bar{O}$ compact.
- (ii) $\mu(O_1) < \mu(O_2)$, if $O_1 \subset O_2$
- (iii) $\mu(O_1 \cup O_2) = \mu(O_1) + \mu(O_2)$,
 if $O_1 \cap O_2 = \emptyset$
- (iv) $\mu(\cup O_i) \leq \sum \mu(O_i)$
- (v) $\mu(O) = \sup \{ \mu(U) \mid U \subset O, \bar{U} \text{ compact} \}$

Then the set function μ^* define by

$$\mu^*(E) = \inf \{ \mu(O) \mid E \subset O \}$$

is a topologically regular outer measure.

348 * Content & Regular content :

Let $\mathcal{K}$ be a family of compact sets containing the compact G 's and having the property that

$$K_1 \cup K_2 \in \mathcal{K}$$

$$\text{and } K_1 \cap K_2 \in \mathcal{K}$$

whenever $K_1 \in \mathcal{K}$ and $K_2 \in \mathcal{K}$.

A non-negative real valued function λ defined on $\mathcal{K}$ is called a content if

- (i) $\lambda(K_1) \leq \lambda(K_2)$ when $K_1 \subset K_2$
- (ii) $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$ when $K_1 \cap K_2 = \emptyset$

It is called a regular content if

- (i) $\lambda(K_1) \leq \lambda(K_2)$ when $K_1 \subset K_2$
- (ii) $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$ when $K_1 \cap K_2 = \emptyset$
- (iii) $\lambda(K) = \inf \{ \lambda(H) : K \subset H^o, H \in \mathcal{K} \}$

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Proposition: Let λ be a content on a class $\mathcal{K}$ of the compact sets, then there is a unique quasi regular Borel measure $\bar{\mu}$ such that for each open set O , we have

$$\bar{\mu}(O) = \sup \{ \lambda(K) \mid K \subset O, K \in \mathcal{K} \}$$

Moreover,

$$\bar{\mu}(K^\circ) \leq \lambda(K) \leq \bar{\mu}(K) \quad \text{for } K \in \mathcal{K}$$

If λ is a regular content,
then $\bar{\mu}(K) = \lambda(K) \quad \forall K \in \mathcal{K}$

Proof:

For each open set O ,
Let $\bar{\mu}(O) = \sup \{ \lambda(K) \mid K \subset O, K \in \mathcal{K} \}$

Now we have,

1. Since λ is non-negative real valued funⁿ.
So for each $K \in \mathcal{K}$

$$\lambda(K) < \infty$$

$$\therefore \bar{\mu}(O) = \sup \{ \lambda(K) \mid K \subset O; K \in \mathcal{K} \} < \infty$$

$$\Rightarrow \bar{\mu}(O) < \infty$$

2.) If $O_1 \subset O_2$

$$\begin{aligned}\text{Then } \bar{\mu}(O_1) &= \sup \{ \lambda(K) \mid K \subset O_1, K \in \mathcal{K} \} \\ &\leq \sup \{ \lambda(K) \mid K \subset O_2, K \in \mathcal{K} \} \\ &\leq \bar{\mu}(O_2)\end{aligned}$$

$$\therefore \text{If } O_1 \subset O_2 \Rightarrow \bar{\mu}(O_1) \leq \bar{\mu}(O_2)$$

3.) If $O_1 \cap O_2 = \emptyset$

Then by definition of content,

$$\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2), \text{ if } K_1 \cap K_2 = \emptyset$$

$$\begin{aligned}\therefore \bar{\mu}(O_1 \cup O_2) &= \sup \{ \lambda(K_1 \cup K_2) \mid K_1 \cup K_2 \subset O_1 \cup O_2, \\ &\quad K_1, K_2 \in \mathcal{K} \} \\ &= \sup \{ \lambda(K_1) + \lambda(K_2) \mid K_1 \subset O_1, K_2 \subset O_2, \\ &\quad K_1, K_2 \in \mathcal{K} \} \\ &= \sup \{ \lambda(K_1) \mid K_1 \subset O_1, K_1 \in \mathcal{K} \} \\ &\quad + \sup \{ \lambda(K_2) \mid K_2 \subset O_2, K_2 \in \mathcal{K} \}\end{aligned}$$

$$\Rightarrow \bar{\mu}(O_1 \cup O_2) = \bar{\mu}(O_1) + \bar{\mu}(O_2), \text{ if } O_1 \cap O_2 = \emptyset$$

4.) Let $O = \bigcup_{i=1}^{\infty} O_i$

and we take a compact set $K \in \mathcal{K}$ with $K \subset O$.

Then there are non-negative real valued continuous functions $\phi_1, \phi_2, \dots, \phi_n$ such that

$$\sum_{i=1}^n \phi_i = 1 \quad \text{on } K$$

and support of $\phi_i \subset O_i$

Let $G_i = \{x : \phi_i(x) \geq \frac{1}{n}\}$

Then each G_i is a compact G_δ set and thus in $\mathcal{K}$.

Consequently, each $G_i \cap K \in \mathcal{K}$

Also we have

$$K = \bigcup_{i=1}^{\infty} (G_i \cap K) \subset \bigcup O_i = O$$

and $G_i \cap K \subset O_i$

$$\begin{aligned} \text{Thus } \lambda(K) &\leq \sum_{i=1}^{\infty} \lambda(G_i \cap K) \\ &\leq \sum_{i=1}^{\infty} \lambda(O_i) \\ &= \sum_{i=1}^{\infty} \mu(O_i) \\ &= \sum_{i=1}^{\infty} \mu(O_i) \end{aligned}$$

$\forall K \subset O$

Therefore taking supremum over all $K \subset O$ such that $K \in \mathcal{K}$,

$$\Rightarrow \sup \{ \lambda(K) \mid K \subset O, K \in \mathcal{K} \} \leq \sum_{i=1}^{\infty} \mu(O_i)$$

$$\Rightarrow \mu(O) \leq \sum_{i=1}^{\infty} \mu(O_i)$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^{\infty} O_i\right) \leq \sum_{i=1}^{\infty} \mu(O_i)$$

5.] By definition of μ ,

$$\mu(O) = \sup \{ \mu(K) \mid K \subset O, K \in \mathcal{K} \}$$

Thus we can extend $\bar{\mu}$ such as

$$\mu^*(E) = \inf \{ \bar{\mu}(O) \mid E \subset O \}$$

is topologically regular outer measure.

Now we know that, "If μ^* is topologically regular outer measure on X , then each Borel set is μ^* -measurable".

This shows that

$$\mu^*(E) = \inf \{ \bar{\mu}(O) \mid E \subset O \} < \infty$$

$$\Rightarrow \bar{\mu}(O) < \infty$$

$$\Rightarrow \bar{\mu} \text{ is Borel measure.}$$

Also for open set O ,

$$\bar{\mu}(O) = \sup \{ \lambda(K) \mid K \subset O, K \in \mathcal{K} \}$$

$$\Rightarrow \bar{\mu} \text{ is inner regular for open sets}$$

$$\text{and } \bar{\mu}(E) = \inf \{ \bar{\mu}(O) \mid E \subset O; O \text{ is open} \}$$

$$\Rightarrow \bar{\mu} \text{ is outer regular.}$$

Thus we have a quasi regular Borel measure $\bar{\mu}$ such that

$$\bar{\mu}(O) = \sup \{ \lambda(K) \mid K \subset O, K \in \mathcal{K} \}$$

Now since $\bar{\mu}$ is outer regular

$\because \mu(K) = \inf \{ \mu(O) \mid K \subset O, O \text{ is open} \}$

$$\Rightarrow \mu(K) \geq \lambda(K) \quad \text{--- (i)}$$

Also $\mu(K^o) = \sup \{ \lambda(H) \mid H \in \mathcal{K}, H \subset K^o \}$

$$\leq \lambda(H) = \lambda(K) \quad \text{--- (ii)}$$

By (i) & (ii),

$$\mu(K^o) \leq \lambda(K) \leq \mu(K)$$

$\because \lambda$ is regular content.

$$\Rightarrow \mu(K) = \inf \{ \mu(O) \mid K \subset O, O \text{ is open} \}$$

$$\leq \inf \{ \mu(H^o) \mid K \subset H^o, H^o \in \mathcal{K} \}$$

$$\leq \inf \{ \lambda(H) \mid K \subset H, H \in \mathcal{K} \}$$

$$= \lambda(K)$$

$$\Rightarrow \mu(K) \leq \lambda(K) \quad \text{--- (iii)}$$

So from eqn (i) & (iii),

$$\mu(K) = \lambda(K) \quad \forall K \in \mathcal{K}$$

* Positive Linear Functional :- A real-valued linear functional

T on $C(X)$ is said to be a positive if $T(f) \geq 0$, whenever $f \geq 0$.

[12, 18, 15] [17/4] [18/1-5] [19/4] [22/4]

(352) * Riesz - Markov Theorem :- Let X be a locally compact Hausdorff space and T a positive linear functional on $C_c(X)$. Then there is a Borel measure μ on X such that

$$T(f) = \int f d\mu \quad \text{for each } f \in C_c(X).$$

The measure μ may be taken to be quasi regular or to be inner regular. In each of these cases it is then unique.

Proof :-

For each open set O , we define $\bar{\mu}(O)$ by

$$\bar{\mu}(O) = \sup \{ T(f) \mid f \in C_c(X),$$

$$0 \leq f \leq 1, \text{ supp } f \subset O \} \quad \text{--- (i)}$$

Then $\bar{\mu}$ is an extended real-valued function on all open sets.

Now we have,

1.] Since I is positive linear functional
and $0 \leq f \leq 1$
 $\therefore I(f) < \infty$

So $\mu(O) < \infty$

2.] If $O_1 \subset O_2$

$$\begin{aligned}\text{Then } \mu(O_1) &= \sup \{ I(f_1) \mid f_1 \in C_c(X), \\ &\quad 0 \leq f_1 \leq 1, \text{ supp } f_1 \subset O_1 \} \\ &\leq \sup \{ I(f_2) \mid f_2 \in C_c(X), \\ &\quad 0 \leq f_2 \leq 1, \text{ supp } f_2 \subset O_2 \} \\ \Rightarrow \mu(O_1) &\leq \mu(O_2)\end{aligned}$$

$$\therefore \mu(O_1) \leq \mu(O_2) \quad \text{if } O_1 \subset O_2$$

3.] If $O = O_1 \cup O_2$

with $O_1 \cap O_2 = \emptyset$

and $f_i \in C_c(X)$, $0 \leq f_i \leq 1$, $\text{supp } f_i \subset O_i$

Then the function $f = f_1 + f_2$ has $\text{supp } f \subset O$
and $0 \leq f \leq 1$.

$$\text{Thus } I(f_1) + I(f_2) = \mu(O)$$

Taking supremum over all $f_1 \neq f_2$, we have

$$\mu(O) = \mu(O_1) + \mu(O_2)$$

4.) Let $O = \bigcup O_i$

and let f be any function in $C(X)$ with $0 \leq f \leq 1$ and $\text{supp } f \subset O$.

Then there are non-negative functions $\phi_1, \phi_2, \dots, \phi_n$ in $C(X)$ with $\text{supp } \phi_i \subset O_i$ and $\sum_{i=1}^n \phi_i = 1$ on $\text{supp } f$.

Then $f = \sum \phi_i f$, $0 \leq \phi_i \leq 1$ & $\text{supp } (\phi_i f) \subset O_i$

$$\therefore I(f) = \sum_{i=1}^n I(\phi_i f) \leq \sum_{i=1}^n \mu(O_i) \\ \leq \sum_{i=1}^{\infty} \mu(O_i)$$

Taking the supremum over all such f gives

$$\mu(O) \leq \sum_{i=1}^{\infty} \mu(O_i)$$

5.) By definition of μ ,

$$\mu(O) = \sup \{ I(f) \mid f \in C(X), 0 \leq f \leq 1, \\ \text{supp } f \subset O \}$$

Thus μ extends to a quasi regular Borel measure.

Next we have to show that

$$I(f) = \int f \, d\mu \quad \forall f \in C(X).$$

Since f is difference of two non-negative functions in $C(X)$,

it suffices to consider $f \geq 0$.

By linearity we may also take $f \leq 1$.

Choose a bounded open set O with $\text{supp } f \subset O$.

$$\text{Set } O_k = \{x : nf(x) > k-1\}$$

$$\text{and } O_0 = \emptyset$$

$$\text{Then } O_{n+1} = \emptyset$$

$$\text{and } \bar{O}_{k+1} \subset O_k.$$

$$\text{Define } \phi_k = \begin{cases} 1 & \text{in } O_{k+1} \\ nf(x) - k + 1 & \text{in } O_k \sim O_{k+1} \\ 0 & \text{in } \bar{O}_k \end{cases}$$

$$\text{Then } f = \frac{1}{n} \sum_{k=1}^n \phi_k$$

$$\text{We also have } \text{supp } \phi_k \subset \bar{O}_k \subset O_{k-1}$$

$$\text{and } \phi_k \equiv 1 \text{ on } O_{k+1}.$$

$$\text{Thus } \bar{\mu}(O_k) \leq I(\phi_k) \leq \bar{\mu}(O_{k-1}) \quad \text{for } k \geq 1$$

— (ii)

$$\text{Also } \bar{\mu}(O_{k+1}) \leq \int \phi_k d\bar{\mu} \leq \bar{\mu}(O_k) \quad \text{for } k \geq 1$$

— (iii)

From eqn (ii) and (iii),

$$\bar{\mu}(O_0) - \bar{\mu}(O_1) \leq \sum_{k=1}^n (I(\phi_k) - \int \phi_k d\bar{\mu}) \leq \bar{\mu}(O_0) + \bar{\mu}(O_1)$$

Consequently,

$$|I(f) - \int f d\bar{\mu}| \leq \frac{2}{n} \bar{\mu}(O)$$

Since n is arbitrary,

$$I(f) = \int f d\bar{\mu}$$

Now by result, 'Let μ be a Baire measure on X . Then there are complete saturated measures $\bar{\mu}$ and $\underline{\mu}$ defined on a σ -algebra containing the Borel sets with $\bar{\mu}$ quasi regular, $\underline{\mu}$ inner regular and

$$\bar{\mu}(E) = \underline{\mu}(E) = \mu(E)$$

for each σ -bounded Baire set.

Then the measures $\bar{\mu}$ and $\underline{\mu}$ are unique'.

This follows that

$$I(f) = \int f d\mu \quad \forall f \in C(X).$$

and μ is unique.

* Metric Space :- Let X be any non-empty set.

A metric on X is a mapping

$d: X \times X \rightarrow \mathbb{R}$ which satisfies the axioms :

(i) $d(x, x) = 0$

(ii) $d(x, y) = 0 \Rightarrow x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Then the set X together with a metric d is called metric space and is denoted by (X, d) .

Remark :- A function $d: X \times X \rightarrow \mathbb{R}$ which satisfies (i), (iii) & (iv) is called a semi-metric and (X, d) is said to be semi-metric space.

- A metric d is always non-negative. For $x, y \in X$

$$d(x, y) + d(y, z) \geq d(x, z)$$

i.e. $2d(x, y) \geq 0 \Rightarrow d(x, y) \geq 0.$

- If $x, y, x', y' \in X$

then $|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$

- Define $d: X \times X \rightarrow \mathbb{R}$ by

$$d(x, x) = 0 \quad \text{and} \quad d(x, y) = 0 \quad ; \quad x \neq y$$

Then (X, d) is called trivial metric space.

* Cauchy sequence :- Let (X, d) be metric space and

let $\{x_n\}$ be a sequence in it.

Then the seqⁿ $\{x_n\}$ is said to be a Cauchy seqⁿ if for every $\epsilon > 0$, $\exists n_0$ such that

$$d(x_n, x_m) < \epsilon$$

$$\forall n, m \geq n_0$$

* Convergence Sequence :- Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is said to be convergent sequence if for every $\varepsilon > 0$, $\exists n_0$ such that

$$d(x_n, x) < \varepsilon \quad \forall n \geq n_0.$$

* Complete Metric Space :- A metric space (X, d) is said to be complete if every Cauchy sequence in it converges to an element of it.

* Diameter :- Let (X, d) be a metric space and $S \subset X$. Then the diameter of S is denoted by $\text{diam } S$ such as

$$\text{diam } S = \sup \{ d(x, y) : x, y \in S \}$$

- A set $S \subset X$ is said to be bounded if $\text{diam } S$ is finite.

Theorem :- Every convergent sequence in a metric space is a Cauchy sequence, but conversely is not true.

* Inequality :- Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a \geq 0$, $b \geq 0$

Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ iff $a^p = b^q$

* Holder's Inequality :- If p and q are non-negative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $f \in L^p$, $g \in L^q$ then $fg \in L^1$ and

$$\int |fg| \leq \left(\int |f|^p \right)^{1/p} \left(\int |g|^q \right)^{1/q}$$

with equality only when $|f|^p = |g|^q$ almost everywhere.

- Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_1, a_2, \dots, a_n \geq 0$ & $b_1, b_2, \dots, b_n \geq 0$.
Then $\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p \right)^{1/p} \left(\sum_{k=1}^n b_k^q \right)^{1/q}$
also $\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k \right) \max b_k$

* Minkowski's Inequality :- If $p > 1$ & $f, g \in L^p[a, b]$
then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

- Let $p \geq 1$, $a_1, a_2, \dots, a_n \geq 0$ & $b_1, b_2, \dots, b_n \geq 0$
Then $\left(\sum_{k=1}^n (a_k + b_k)^p \right)^{1/p} \leq \left(\sum_{k=1}^n a_k^p \right)^{1/p} + \left(\sum_{k=1}^n b_k^p \right)^{1/p}$

* Schwarz's Inequality :- If $f, g \in L^2[a, b]$
then $f, g \in L[a, b]$ and
 $\|fg\| \leq \|f\|_2 \cdot \|g\|_2$

* Neighbourhood or Open Sphere or Open Ball :-
Let (X, d) be a metric space. Then the set
 $B_r(a) = \{x \in X : d(x, a) < r\}$
is called a neighbourhood with centre a & radius r .

* Topological Space :- Let X be any non empty set.
Then the collection T of subsets of
power set of X is said to be topology if :

- $\emptyset$ and $X \in T$.
- Any union of member of T is also member of T .
- The intersection of finite number of members of T is also member of T .

Thus the set X together with a topology T is
said to be topological space and denoted by
 (X, T) .

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* Metriizable :- A topological space (X, T) is called metriizable if its topology T can be defined by a metric d .

- Every metric space is metriizable.
- Let X be a topological space. Then a set $A \subset X$ is said to be dense in X , if $\bar{A} = X$.
- Let X be a topological space. Then a set $A \subset X$ is said to be nowhere dense, if $\bar{A}$ has empty interior.
- A topological space (X, T) is called sepeerable if X contains a countable dense subset.

* Internal Composition :- Let V be a non-empty set. Then a mapping $f: V \times V \rightarrow V$ in V is said to be internal composition, if

$$\alpha + \beta = \gamma \quad \forall \alpha, \beta, \gamma \in V$$

* External Composition :- Let V and F be a non-empty sets. Then a mapping $f: V \times F \rightarrow V$ in V is said to be external composition, if

$$\alpha \circ \alpha = \beta \quad \forall \alpha, \beta \in V$$

$$\forall a \in F$$

* Vector Space :- Let V be a set, whose elements are called vectors and F be a field, whose elements are called scalar. Then the set V is called vector space or linear space over the field F , if it satisfies the following properties:

1. $(V, +)$ is abelian group w.r. to internal composition '+'.
 2. $V(F)$ is closed under scalar multiplication with respect to external composition '·'.

3. $(V, +, \cdot)$ satisfies that:

- (i) $a\alpha \in V \quad \forall a \in F \text{ \& } \alpha \in V$
- (ii) $a(\alpha + \beta) = a\alpha + a\beta \quad \forall a \in F \text{ \& } \alpha, \beta \in V$
- (iii) $(a+b) \cdot \alpha = a\alpha + b\alpha \quad \forall a, b \in F \text{ \& } \alpha \in V$
- (iv) $a(b \cdot \alpha) = (a \cdot b)\alpha \quad \forall a, b \in F \text{ \& } \alpha \in V$
- (v) $1 \cdot \alpha = \alpha \quad \forall \alpha \in V \text{ \& } 1 \in F$

* Vector sub-space :- Let V be a vector space over a field F and $W \subseteq V$. Then W is called a subspace of V , if

- (i) $x, y \in W \Rightarrow x + y \in W$
- (ii) $x \in W, \lambda \in F \Rightarrow \lambda x \in W$

* Quotient Space :- Let X be a vector space and M be a subspace of X . Define a relⁿ '~' in X such that for $x, y \in X$

$$x \sim y \Leftrightarrow x - y \in M \quad \text{where } x - y = x + (-y).$$

Then the set of all equivalence class '~' is defined by

$$X/M = \{x + M : x \in X\}$$

and for $x, y \in X$ and $\lambda \in F$

$$(i) (x + M) + (y + M) = (x + y) + M$$

$$(ii) \lambda(x+M) = \lambda x + M$$

So the vector space X/M is called the quotient space of X by M .

★ Linear Combination :- Let V be a vector space over a field F and let $\lambda_1, \lambda_2, \dots, \lambda_n \in F$. Then an element $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$ is called a linear combination of the elements $x_1, x_2, \dots, x_n$ of X and also written as $\sum_{i=1}^n \lambda_i x_i$

★ Linearly Dependent :- Let $V(F)$ be a vector space. Then $x_1, x_2, \dots, x_n \in V$ is said to be linearly dependent if $\exists$ scalars $a_1, a_2, \dots, a_n \in F$ such that not all of them are zero for

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

★ Linearly Independent :- Let $V(F)$ be a vector space. Then $x_1, x_2, \dots, x_n \in V$ is said to be linearly independent if $\exists$ $a_1, a_2, \dots, a_n \in F$ such that

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

$\Rightarrow a_i = 0 \quad ; i = 1, 2, \dots, n.$

★ Linear Transformation :- Let U and V be any two vector spaces over a field F . Then a mapping $T: U(F) \rightarrow V(F)$ is said to be linear transformation, if

$$(i) T(a+b) = T(a) + T(b) \quad \forall a, b \in U$$

$$(ii) T(\alpha \cdot a) = \alpha \cdot T(a) \quad \forall a \in U, \alpha \in F$$

Exa.: • The zero mapping $0: X \rightarrow Y$ defined by

$$0(x) = 0 \quad \forall x \in X$$
 is a trivial linear transformation.

• The identity mapping $\text{id}: X \rightarrow X$ defined by

$$\text{id}(x) = x \quad \forall x \in X$$
 is a linear transformation.

• A constant map $f: R \rightarrow R$ defined by

$$f(x) = a \quad \text{where } a \in R \text{ is fixed}$$
 is not a linear transformation, because

$$f(x+y) = a \neq f(x) + f(y) = a+a.$$

* Linear Operator :- Let X be a vector space. Then a linear transformation map $f: X \rightarrow X$ is called a linear operator.

* Linear Functionals :- Let X be a complex or real vector space. Then a mapping $f: X \rightarrow C$ or R is said to be linear functional if

$$f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$$

$$\forall x_1, x_2 \in X \text{ and } c_1, c_2 \in C \text{ or } R.$$

* Sub-linear Functionals :- Let X be a real vector space. Then a real-valued funⁿ p defined on X is called a sub-linear functional, if

$$p(x+y) \leq p(x) + p(y)$$
 and

$$p(\alpha x) = \alpha \cdot p(x) \quad \forall x, y \in X \text{ \& } \alpha \in F$$

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(44) ★ Normed Linear Space :- Let X be a complex or real vector space of a finite or infinite dimension. Let K be the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

A norm on X is a real function $\|\cdot\|: X \rightarrow K$ defined on X such that for any $x, y \in X$ and $\forall \lambda \in K$,

$$(i) \quad \|x\| \geq 0$$

$$(ii) \quad \|x+y\| \leq \|x\| + \|y\|$$

$$(iii) \quad \|\lambda x\| = |\lambda| \cdot \|x\|$$

$$(iv) \quad \|x\| = 0 \Rightarrow x = 0$$

Then $\|\cdot\|$ is called norm and X together with norm i.e. $(X, \|\cdot\|)$ is called a normed linear space.

Remark :- In condition (iii), if $\lambda = 0$, then $\|0\| = 0$

So by condⁿ (iv), $\|x\| = 0$ iff $x = 0$

• In condition (iii), if $\lambda = -1$, then $\|-x\| = \|x\|$

In particular, $\|y-x\| = \|x-y\|$

• We note that,

$$\|x\| - \|y\| \leq \|x-y\| \leq \|x\| + \|-y\| = \|x\| + \|y\| \quad \text{--- (i)}$$

Now we replace x by y and y by x such that

$$\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$$

$$\Rightarrow \|x\| - \|y\| \geq -\|x-y\| \quad \text{--- (ii)}$$

From eqⁿ (i) & (ii), we have

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$$

$$\Rightarrow |\|x\| - \|y\|| \leq \|x-y\|$$

Note :- Normed linear space is a vector space with a norm.

(44) * Semi Norm :- Let X be a real or complex vector space and K be the field of R or C . If a norm on X is a real funⁿ $\|\cdot\|: X \rightarrow K$ defined on X such that for any $x, y \in X$ & $\lambda \in K$,

- (i) $\|x\| \geq 0$
- (ii) $\|x+y\| \leq \|x\| + \|y\|$
- (iii) $\|\lambda x\| = |\lambda| \cdot \|x\|$

Then the norm $\|\cdot\|$ is called a seminorm.

(45) * p-Norm :- A norm on X is a real function $\|\cdot\|: X \rightarrow R$ or C define on X such that for any $x, y \in X$ and $\lambda \in K$,

- (i) $\|x\| \geq 0$
- (ii) $\|x+y\| \leq \|x\| + \|y\|$
- (iii) $\|\lambda x\| = |\lambda|^p \cdot \|x\|$
- (iv) $\|x\| = 0 \Rightarrow x = 0$

Then the norm $\|\cdot\|$ is called p-norm.

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(46) * Examples of Normed Linear Space :-

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1. R^n or C^n is a normed linear space with the norm defined by $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}$

Solⁿ :-

Let $X = R^n$ or C^n be a space and $x \in X$ is defined by

$$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}$$

- (i) Since $|x_i| \geq 0$; $i = 1, 2, \dots, n$
- $\Rightarrow |x_i|^2 \geq 0$; $i = 1, 2, \dots, n$
- $\Rightarrow \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \geq 0 \Rightarrow \boxed{\|x\|_2 \geq 0}$

- l_p -space: Seqⁿ of $\{x_n\}$ such that $\sum |x_n|^p < \infty$
OR $\{x_n\} : \sum |x_n|^p < \infty$
- l_∞ -space: $\{x_n\} : \sup |x_n|^p < \infty$ OR $\{x_n\} : \max |x_n|^p < \infty$

$$\text{(ii)} \quad \|x+y\|_2 = \left(\sum_{i=1}^n |x_i+y_i|^2 \right)^{1/2} \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} + \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2}$$

$$\Rightarrow \boxed{\|x+y\|_2 \leq \|x\|_2 + \|y\|_2} \quad [\text{by Minkowski inequality}]$$

$$\text{(iii)} \quad \|\lambda x\|_2 = \left(\sum_{i=1}^n |\lambda x_i|^2 \right)^{1/2} = |\lambda|^{2 \times \frac{1}{2}} \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}$$

$$\Rightarrow \boxed{\|\lambda x\|_2 = |\lambda| \cdot \|x\|_2}$$

$$\text{(iv)} \quad \|x\|_2 = 0 \Rightarrow \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} = 0 \Rightarrow x = 0$$

$$\Rightarrow \boxed{\|x\|_2 = 0 \Rightarrow x = 0}$$

Thus R^n or C^n is normed linear space.

2/ l_∞ is normed linear space with the norm defined by

$$\|x\| = \sup_k |x_k|$$

Solⁿ: Let $X = l_\infty$ be a space

and $x \in X$ is defined by $\|x\| = \sup_k |x_k|$

$$\text{(i)} \quad \text{Since } |x_k| \geq 0$$

$$\Rightarrow \sup_k |x_k| \geq 0 \Rightarrow \boxed{\|x\| \geq 0}$$

$$\text{(ii)} \quad \text{Since } |x_k + y_k| \leq |x_k| + |y_k|$$

$$\Rightarrow \sup_k |x_k + y_k| \leq \sup_k [|x_k| + |y_k|] = \sup_k |x_k| + \sup_k |y_k|$$

$$\Rightarrow \boxed{\|x+y\| \leq \|x\| + \|y\|}$$

$$\text{(iii)} \quad \|\lambda x\| = \sup_k |\lambda x_k| = |\lambda| \sup_k |x_k| = |\lambda| \cdot \|x\|$$

$$\Rightarrow \boxed{\|\lambda x\| = |\lambda| \cdot \|x\|}$$

$$\text{(iv)} \quad \|x\| = 0 \Rightarrow \sup_k |x_k| = 0 \Rightarrow x = 0$$

$$\Rightarrow \boxed{\|x\| = 0 \Rightarrow x = 0}$$

Thus l_∞ is a normed linear space.

(45)

Proposition: Every nls X is metric space, relative to the natural metric d defined by

$$d(x, y) = \|x - y\|$$

Proof: Let X be a normed linear space and a function $\|\cdot\|: X \rightarrow \mathbb{R}$ defined on X such that

$$(i) \quad \|x\| \geq 0$$

$$(iii) \quad \| \lambda x \| = |\lambda| \cdot \|x\|$$

$$(ii) \quad \|x + y\| \leq \|x\| + \|y\|$$

$$(iv) \quad \|x\| = 0 \Rightarrow x = 0$$

Then we have to show that X is metric space.

For this,

$$(1.) \quad \text{By definition, } d(x, y) = \|x - y\| \geq 0 \quad [\text{by (i)}]$$

$$\Rightarrow d(x, y) \geq 0$$

$$(2.) \quad d(x, y) = 0 \Rightarrow \|x - y\| = 0 \Rightarrow x = y \quad [\text{by (iv)}]$$

$$\Rightarrow d(x, y) = 0 \Leftrightarrow x = y$$

$$(3.) \quad d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$$

$$\Rightarrow d(x, y) = d(y, x)$$

$$(4.) \quad d(x, z) = \|x - z\| = \|x - y + y - z\| \leq \|x - y\| + \|y - z\| \quad [\text{by (ii)}]$$

$$\Rightarrow d(x, z) \leq d(x, y) + d(y, z)$$

Thus by (1.), (2.), (3.) and (4.), we obtain
 X is metric space.

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Proposition: The vector space $X \neq \{0\}$ with the discrete metric defined by

$$d(x, y) = \begin{cases} 0 & , \text{ if } x = y \\ 1 & , \text{ if } x \neq y \end{cases} \quad \forall x, y \in X$$

is a metric space but not a normed linear space.

Proof:

Let X be a metric space.

and a function $d: X \times X \rightarrow \mathbb{R}$ is defined by

$$d(x, y) = \|x - y\| \quad \forall x, y \in X$$

such that

(i) $d(x, y) \geq 0$

(iii) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0 \Rightarrow x = y$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Then we have to show that X is not nls.

For this,

(1) By defⁿ, $\|x - y\| = d(x, y) \geq 0$ [by (i)]

$\Rightarrow \|x - y\| \geq 0$

(2) Let $\|P\| = \|x - y\|$ and $\|Q\| = \|y - z\|$

$\therefore \|P + Q\| = \|x - y\| + \|y - z\| = \|x - y + y - z\| = \|x - z\|$

By (iv), we have

$$d(x, z) \leq d(x, y) + d(y, z)$$

$$\Rightarrow \|x - z\| \leq \|x - y\| + \|y - z\|$$

$$\Rightarrow \|P + Q\| \leq \|P\| + \|Q\|$$

(3) If $y = 0$, then $d(x, 0) = \|x - 0\| = \|x\| = 1$ [by defⁿ]

• If $\lambda = 0$, then $\|\lambda x\| = 0$ and $|\lambda| \cdot \|x\| = 0$

$$\Rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|$$

• If $\lambda = -1$, then $\|\lambda x\| = \|-x\| = 1 \cdot \|x\| = |-1| \cdot \|x\| = |\lambda| \cdot \|x\|$

$$\Rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|$$

• If $\lambda = 1$, then $\|\lambda x\| = \|x\| = 1 \cdot \|x\| = |1| \cdot \|x\| = |\lambda| \cdot \|x\|$

$$\Rightarrow \|\lambda x\| = |\lambda| \cdot \|x\|$$

• If $\lambda \neq \pm 1$, so let $\lambda = 2$

then $\|\lambda x\| = d(\lambda x, 0) = 1$

- $C[a, b]$ = set of all continuous function from $[a, b]$.

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$$\text{and } |\lambda| \cdot \|x\| = |\lambda| \cdot d(x, 0) = |\lambda|$$

$$\text{So } \|\lambda x\| = |\lambda| \cdot \|x\|$$

$$\Rightarrow |\lambda| = 1$$

which is contradict that $\lambda \neq \pm 1$

Thus X is not a nls.

Example: $C[a, b]$ is normed linear space with the norm defined by

$$\|f\| = \max_{a \leq x \leq b} |f(x)|$$

Solⁿ:

Let $X = C[a, b]$ be a space

and $f \in X$ is defined by

$$\|f\| = \max_{a \leq x \leq b} |f(x)|$$

$$(1.) \text{ Since } |f(x)| \geq 0 \Rightarrow \max_{a \leq x \leq b} |f(x)| \geq 0 \\ \Rightarrow \|f\| \geq 0$$

$$(2.) \|f+g\| = \max_{a \leq x \leq b} |(f+g)(x)| = \max_{a \leq x \leq b} |f(x)+g(x)| \\ \leq \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} |g(x)| \\ \Rightarrow \|f+g\| \leq \|f\| + \|g\|$$

$$(3.) \|\lambda \cdot f\| = \max_{a \leq x \leq b} |\lambda f(x)| = |\lambda| \cdot \max_{a \leq x \leq b} |f(x)| \\ \Rightarrow \|\lambda \cdot f\| = |\lambda| \cdot \|f\|$$

$$(4.) \|f\| = 0 \Rightarrow \max_{a \leq x \leq b} |f(x)| = 0 \Rightarrow |f(x)| = 0 \quad \forall x \in [a, b] \\ \Rightarrow \|f\| = 0 \Rightarrow f = 0$$

Thus $C[a, b]$ is normed linear space.

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Example: R^n or C^n is a normed linear space with the norm defined by

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

Solⁿ:

Let $X = R^n$ or C^n be a space.

and $x \in X$ is defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$

(1.) Since $|x_i| \geq 0$; $i = 1, 2, \dots, n$

$$\Rightarrow \sum_{i=1}^n |x_i| \geq 0 \quad \Rightarrow \quad \|x\|_1 \geq 0$$

(2.) $\|x+y\|_1 = \sum_{i=1}^n |x_i+y_i| \leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i|$

$$\Rightarrow \|x+y\|_1 \leq \|x\|_1 + \|y\|_1$$

(3.) $\|\lambda x\|_1 = \sum_{i=1}^n |\lambda x_i| = |\lambda| \cdot \sum_{i=1}^n |x_i|$

$$\Rightarrow \|\lambda x\|_1 = |\lambda| \cdot \|x\|_1$$

(4.) $\|x\|_1 = 0 \Rightarrow \sum_{i=1}^n |x_i| = 0 \Rightarrow |x_i| = 0$; $\forall i = 1, 2, \dots, n$

$$\Rightarrow \|x\|_1 = 0 \Rightarrow x = 0$$

Thus R^n or C^n is normed linear space.

Example: R^n or C^n is a normed linear space with the norm defined by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Example: c and c_0 are normed linear space with the norm defined by

$$\|x\| = \sup |x_n|$$

where $c = \mathbb{R}$ is an space of convergent sequence.

and $c_0 = \mathbb{R}$ is an space of null sequence.

Example: In C if we define $\|x\| = \lim_{n \rightarrow \infty} |x_n|$, then it is not a nls. However, this is a seminorm on C .

Solⁿ:

Let $X = C$ be a space and $x \in X$ is defined by
$$\|x\| = \lim_{n \rightarrow \infty} |x_n|$$

(1.) Since $|x_n| \geq 0 \Rightarrow \lim_{n \rightarrow \infty} |x_n| \geq 0$
 $\Rightarrow \|x\| \geq 0$

(2.) $\|x+y\| = \lim_{n \rightarrow \infty} [|x_n + y_n|] \leq \lim_{n \rightarrow \infty} |x_n| + \lim_{n \rightarrow \infty} |y_n|$
 $\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

(3.) $\|\lambda \cdot x\| = \lim_{n \rightarrow \infty} |\lambda x_n| = |\lambda| \cdot \lim_{n \rightarrow \infty} |x_n|$
 $\Rightarrow \|\lambda \cdot x\| = |\lambda| \cdot \|x\|$

(4.) Take $x_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$
 $\therefore \|x\| = 0 \Rightarrow \lim_{n \rightarrow \infty} |x_n| = 0$
 $\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{1}{n} \right| = 0$
 $\Rightarrow \frac{1}{n} = 0 \quad \forall n \in \mathbb{N}$
which is not possible.

Thus C is not a normed linear space
but C is a seminorm space.

(46) * Definitions :- Let X be a normed linear space

- The open ball $B_r(a)$ with centre $a \in X$ and radius $r \geq 0$ is the set of all points $x \in X$ such that
 $d(a, x) < r$ or $\|x - a\| < r$.

- The closed ball $\bar{B}_r(a)$ with centre $a \in X$ and radius $r \geq 0$ is the set of all points $x \in X$ such that
 $d(x, a) \leq r$ or $\|x - a\| \leq r$
- The sphere $S_r(a)$ with centre $a \in X$ and radius $r \geq 0$ is the set of all points $x \in X$ such that
 $d(a, x) = r$ or $\|x - a\| = r$
- Let $r = 1$ and $a = 0$, then
 Unit open ball $\Rightarrow d(x, 0) < 1$ or $\|x\| < 1$
 Unit closed ball $\Rightarrow d(x, 0) \leq 1$ or $\|x\| \leq 1$
 Unit sphere $\Rightarrow d(x, 0) = 1$ or $\|x\| = 1$

Proposition: Let X be a nls with the norm $d(x, y) = \|x - y\|$.
 Then for any $x, y, z \in X$ and $\forall \lambda \in K$,

- $d(0, x) = \|x\|$
- $d(x+z, y+z) = d(x, y)$
- $d(\lambda x, \lambda y) = |\lambda| \cdot d(x, y)$

Proof:

1) Since $d(x, y) = \|x - y\|$

Taking $y = 0$, then $d(x, 0) = \|x - 0\| = \|x\|$
 $\Rightarrow d(0, x) = \|x\|$

2) $d(x+z, y+z) = \|x+z - (y+z)\| = \|x - y\|$
 $\Rightarrow d(x+z, y+z) = d(x, y)$

3) $d(\lambda x, \lambda y) = \|\lambda x - \lambda y\| = |\lambda| \cdot \|x - y\|$
 $\Rightarrow d(\lambda x, \lambda y) = |\lambda| \cdot d(x, y)$

Theorem: If $(X, \|\cdot\|)$ is a normed linear space, then the vector
 $\|\cdot\|: X \rightarrow \mathbb{R}$ is a continuous function on X .

Proof:

Let $(X, \|\cdot\|)$ is a normed linear space.

Then we have to show that $\|\cdot\|: X \rightarrow \mathbb{R}$ is a continuous function on X .

For this, let $x_n \rightarrow x$ in X ,

then we only need to show that $\|x_n\| \rightarrow \|x\|$ in $\mathbb{R}$.

Since $x_n \rightarrow x$ in X

Therefore for a given $\varepsilon > 0$, $\exists n_0$ such that

$$\|x_n - x\| < \varepsilon \quad \forall n \geq n_0 \quad \text{--- (i)}$$

$$\text{Also, } |\|x_n\| - \|x\|| \leq \|x_n - x\| < \varepsilon \quad \forall n \geq n_0 \quad \text{[by (i)]}$$

$$\Rightarrow |\|x_n\| - \|x\|| < \varepsilon \quad \forall n \geq n_0$$

$$\Rightarrow \|x_n\| \rightarrow \|x\| \text{ in } \mathbb{R}$$

$$\Rightarrow \|\cdot\| \text{ is a continuous function on } X.$$

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Proposition: Let X be a nls. Then the mappings

$$1.) (x, y) \in X \times X \rightarrow x+y \in X \quad [18/2.5]$$

$$2.) (\lambda, x) \in K \times X \rightarrow \lambda x \in X$$

$$3.) (x, y) \in X \times X \rightarrow d(x, y) = \|x - y\| \in \mathbb{R}$$

$$4.) x \in X \rightarrow \|x\| \in \mathbb{R} \quad \text{are all continuous.}$$

Proof:

Let X be a normed linear space.

1.) Let $(a, b) \in X \times X$ be arbitrary point,
 then $a+b$ is its image.

Then we have to show that the mapping is continuous.

So for a given $\varepsilon > 0$, $\exists \delta > 0$ such that

$x \in B_\delta(a)$ & $y \in B_\delta(b)$ i.e. $\|x-a\| < \delta$ & $\|y-b\| < \delta$

Therefore,

$$\begin{aligned}\|(x+y)-(a+b)\| &= \|(x-a)+(y-b)\| \leq \|x-a\| + \|y-b\| \\ &< \delta + \delta = 2\delta = \varepsilon \text{ (say)}\end{aligned}$$

$$\Rightarrow \|(x+y)-(a+b)\| < \varepsilon$$

$\Rightarrow$ The mapping is continuous.

2.) Let $(\alpha, a) \in K \times X$ be arbitrary point,

so αa is its image.

Then we have to show that the mapping is continuous.

So for a given $\varepsilon > 0$, $\exists \delta > 0$ such that

$|\lambda - \alpha| < \delta$ and $x \in B_\delta(a)$ i.e. $\|x-a\| < \delta$

Therefore,

$$\begin{aligned}\|\lambda x - \alpha a\| &= \|\lambda x - \alpha a + \lambda a - \lambda a\| \leq |\lambda| \cdot \|x-a\| + \|a\| \cdot |\lambda - \alpha| \\ &< |\lambda| \cdot \delta + \|a\| \cdot \delta = \varepsilon \text{ (say)}\end{aligned}$$

$$\Rightarrow \|\lambda x - \alpha a\| < \varepsilon$$

$\Rightarrow$ The mapping is continuous.

3.) Let $(a, b) \in X \times X$ be arbitrary point.

so $d(a, b) = \|b-a\|$ is its image.

Then we have to show that the mapping is continuous.

So for a given $\varepsilon > 0$, $\exists \delta > 0$ such that

$x \in B_\delta(a)$ & $y \in B_\delta(b)$ i.e. $\|x-a\| < \delta$ & $\|y-b\| < \delta$

Therefore,

$$|\|y-x\| - \|b-a\|| \leq \|(y-x)-(b-a)\| = \|(y-b)-(x-a)\|$$

$$\Rightarrow \|y-x\| - \|b-a\| \leq \|y-b\| + \|x-a\| < \delta + \delta = 2\delta = \varepsilon \text{ (say)}$$

$$\Rightarrow \|y-x\| - \|b-a\| < \varepsilon$$

$\Rightarrow$ The mapping is continuous.

4. Let $a \in X$ be arbitrary point.

So $\|a\|$ is its image.

Then we have to show that the mapping is continuous.

So for a given $\varepsilon > 0$, $\exists \delta > 0$ such that $\|x-a\| < \delta$ i.e. $x \in B(a, \delta)$

Therefore,

$$\| \|x\| - \|a\| \| \leq \|x-a\| < \delta = \varepsilon \text{ (say)}$$

$$\Rightarrow \| \|x\| - \|a\| \| < \varepsilon$$

$\Rightarrow$ The mapping is continuous.

★. Homeomorphism :- Let X and Y be two normed linear spaces. Then the mapping

$f: X \rightarrow Y$ is called Homeomorphism if

1. f is one-one.

2. f is onto.

3. f is continuous

4. f^{-1} is continuous or f is open.

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Corollary: Let X be a nls. Let $a \in X$ and $\alpha \in K$, $\alpha \neq 0$. Then the mapping $x \in X \rightarrow \alpha x + a \in X$ is a homeomorphism of X into itself.

Proof: Let X be a normed linear space.

and $f: X \rightarrow X$ such that

$$f(x) = \alpha x + a$$

$$\forall x, a \in X \text{ } \& \alpha \in K, \alpha \neq 0$$

Then we have to show that f is homeomorphism.

- To show - f is one-one :

Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

$$\Rightarrow \alpha x_1 + a = \alpha x_2 + a$$

$$\Rightarrow x_1 = x_2$$

= Thus f is one-one.

- To show - f is onto :

If $x \in X$, then $f(x) = \alpha x + a \in X$

Let $f(x) = y$, then $y = \alpha x + a$

$$\Rightarrow y - a = \alpha x$$

$$\Rightarrow \alpha^{-1}(y - a) = x \in X$$

$$\therefore f(\alpha^{-1}(y - a)) = \alpha(\alpha^{-1}(y - a)) + a = y - a + a = y \in X$$

Thus f is onto.

- To show - f is continuous : Let $b \in X$

Suppose for a given $\epsilon > 0$, $\exists \delta > 0$ s.t. $x \in B_\delta(b)$ i.e. $\|x - b\| < \delta$

$$\begin{aligned} \therefore \|f(x) - f(b)\| &= \|(\alpha x + a) - (\alpha b + a)\| = \|\alpha(x - b)\| \\ &= |\alpha| \cdot \|x - b\| < |\alpha| \cdot \delta = \epsilon \text{ (say)} \end{aligned}$$

$$\Rightarrow \|f(x) - f(b)\| < \epsilon$$

Thus f is continuous.

- To show - f^{-1} is continuous :

For $x \in X$, we have $f(x) = \alpha x + a \Rightarrow x = f^{-1}(\alpha x + a)$.

Let $b \in X$ and for a given $\epsilon > 0$, $\exists \delta > 0$ such that $x \in B_\delta(b)$ i.e. $\|x - b\| < \delta$

$$\therefore \|f^{-1}(\alpha x + a) - f^{-1}(\alpha b + a)\| = \|x - b\| < \delta = \epsilon \text{ (say)}$$

Thus f^{-1} is continuous

Hence f is homeomorphism.

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Theorem: A normed linear space X is complete iff every absolutely convergent series in X is convergent.

Proof:

Let X is complete nls.

Then we have to show that every absolutely convergent series in X is convergent.

Let $\sum_{n=1}^{\infty} x_n$, $x_n \in X$ be an absolutely convergent series.
Then we have, $\sum_{n=1}^{\infty} \|x_n\| < \infty$

Let y_k be the k^{th} partial sum of the series $\sum_{n=1}^{\infty} x_n$
i.e. $y_k = \sum_{n=1}^k x_n$

$$\begin{aligned} \text{Now } \|y_{k+p} - y_k\| &= \left\| \sum_{n=1}^{k+p} x_n - \sum_{n=1}^k x_n \right\| = \left\| \sum_{n=k+1}^{k+p} x_n \right\| \\ &= \|x_{k+1} + x_{k+2} + \dots + x_{k+p}\| \\ &\leq \|x_{k+1}\| + \|x_{k+2}\| + \dots + \|x_{k+p}\| \\ &= \sum_{n=k+1}^{k+p} \|x_n\| < \infty \end{aligned}$$

$$\text{i.e. } \|y_{k+p} - y_k\| \leq \sum_{n=k+1}^{k+p} \|x_n\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

Thus $\{y_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X .

Since X is complete.

Therefore $\exists x \in X$ such that

$$\begin{aligned} x &= \lim_{k \rightarrow \infty} y_k \\ \Rightarrow x &= \lim_{k \rightarrow \infty} \sum_{n=1}^k x_n \\ \Rightarrow x &= \sum_{n=1}^{\infty} x_n \end{aligned}$$

Hence the series $\sum_{n=1}^{\infty} x_n$ is convergent.

Conversely, suppose every absolutely convergent series in X is convergent.

Then we have to show that X is complete nls.

For this, let $\{x_n\}$ be a Cauchy sequence in X .

Then for each positive integer k , $\exists$ a positive integer n_k such that

$$\|x_n - x_m\| < \frac{1}{2^k} \quad \forall n, m \geq n_k$$

Choose $n_{k+1} > n_k$

and $y_1 = x_{n_1}$ and $y_{k+1} = x_{n_{k+1}} - x_{n_k}$; $k \geq 1$

Then

$$\begin{aligned} \sum_{k=1}^{\infty} \|y_k\| &= \|y_1\| + \sum_{k=2}^{\infty} \|y_k\| = \|x_{n_1}\| + \sum_{k=2}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| \\ &< \|x_{n_1}\| + \sum_{k=2}^{\infty} \frac{1}{2^k} = \|x_{n_1}\| + \frac{1/2}{1-1/2} \\ &= \|x_{n_1}\| + 1 < \infty \end{aligned}$$

$$\Rightarrow \sum_{k=1}^{\infty} \|y_k\| < \infty$$

$$\Rightarrow \sum_{k=1}^{\infty} \|y_k\| \text{ is convergent.}$$

$$\Rightarrow \sum_{k=1}^{\infty} y_k \text{ is absolutely convergent in } X.$$

So by our assumption, every absolutely convergent series in X is convergent.

Then $\exists y \in X$ such that

$$\begin{aligned} y &= \lim_{n \rightarrow \infty} \sum_{k=1}^n y_k = \lim_{n \rightarrow \infty} [y_1 + y_2 + \dots + y_n] \\ \Rightarrow y &= \lim_{n \rightarrow \infty} [x_{n_1} + (x_{n_2} - x_{n_1}) + \dots + (x_{n_n} - x_{n_{n-1}})] \\ \Rightarrow y &= \lim_{n \rightarrow \infty} x_{n_n} \end{aligned}$$

Thus $\{x_{n_n}\}$ i.e. $\{x_n\}$ is convergent to y in X i.e. Cauchy sequence $\{x_n\}$ is convergent in X .

Hence X is complete nls.

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Theorem⁽¹¹⁹⁾: Let M be a closed linear subspace of a nls X . Then the quotient space X/M is a nls with the norm

$$\|x+M\| = \inf \{\|x+m\| : m \in M\} \quad \text{for } x \in X$$

Further, if X is complete nls or Banach space, then so is X/M .

Proof : Let M be a closed linear subspace of nls X .
Then firstly we have to show that X/M is a
nls with the norm

$$\|x+M\| = \inf \{ \|x+m\| : m \in M \}$$

1. $\|x+M\| \geq 0 \quad \forall x+M \in X/M$

2. $\|x+M\| = 0 \Leftrightarrow x+M = M$ i.e. M is a zero element of X/M

Since M is closed.

Then $\exists$ a sequence $\{m_k\}_{k=1}^{\infty} \subset M$ such that

$$\|x+m_k\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

$$\text{i.e. } \lim_{k \rightarrow \infty} \|x+m_k\| = 0$$

$$\Rightarrow \|x + \lim_{k \rightarrow \infty} m_k\| = 0$$

$$\Rightarrow x + \lim_{k \rightarrow \infty} m_k = 0$$

$$\Rightarrow \lim_{k \rightarrow \infty} m_k = -x$$

$$\Rightarrow -x \in M$$

Since $x \in X$ and $M \subset X \Rightarrow x \in M$

Therefore, $x+M = M$ or $x-x = 0$

$$\Rightarrow \|x+M\| = 0$$

or $\|x+M\| = M$, where M is zero element of X/M

3. $\|(x+M)+(y+M)\| = \|(x+y)+M\|$ [By property of coset]
 $= \inf \{ \|(x+y)+m\| : m \in M \}$
 $= \inf \{ \|x+M+y+m'\| : m, m' \in M \}$
 $\leq \inf \{ \|x+m\| : m \in M \} + \inf \{ \|y+m'\| : m' \in M \}$
 $\Rightarrow \|(x+M)+(y+M)\| \leq \|x+M\| + \|y+M\|$

4. $\|\lambda \cdot (x+M)\| = \|\lambda x+M\| = \inf \{ \|\lambda x+m\| : m \in M \}$

$$\begin{aligned}\Rightarrow \|\lambda \cdot (x+M)\| &= \inf \{ |\lambda| \cdot \|x+m\| : m \in M \} \\ &= |\lambda| \cdot \inf \{ \|x+m\| : m \in M \} \\ \Rightarrow \|\lambda \cdot (x+M)\| &= |\lambda| \cdot \|x+M\|\end{aligned}$$

Thus X/M is normed linear space.

Further, let X is complete nls or Banach space.
Then we prove that X/M is complete nls.

Let $\{y_k+M\}$ be a Cauchy sequence in X/M
So $\{y_k\} \in X$

Since $\{y_k+M\}$ is a Cauchy sequence in X/M .

Then for $\varepsilon = \frac{1}{2}$, $\exists$ +ve integer n_1 such that

$$\|(y_n+M) - (y_m+M)\| < \frac{1}{2} \quad \forall m, n \geq n_1$$

$\therefore$ Set $y_{n_1} = x_1 \in X$.

Similarly for $\varepsilon = \frac{1}{2^2}$, $\exists$ +ve integer n_2 such that

$$\|(y_n+M) - (y_m+M)\| < \frac{1}{2^2} \quad \forall m, n \geq n_2$$

$\therefore$ Set $y_{n_2} = x_2 \in X$

Continuing this way, we get $y_{n_k} = x_k$

$$\text{and } \|(y_n+M) - (y_m+M)\| < \frac{1}{2^k} \quad \forall n, m \geq n_k$$

$$\text{or } \|(x_k+M) - (x_{k+1}+M)\| < \frac{1}{2^k} \quad \forall k = 1, 2, \dots$$

Let $z_k = x_k + M \quad ; k = 1, 2, \dots$

$$\text{Then } \|z_1 - z_2\| < \frac{1}{2}$$

$$\|z_2 - z_3\| < \frac{1}{2^2}$$

$\vdots$

$$\|z_k - z_{k+1}\| < \frac{1}{2^k}$$

If $m < n$

$$\begin{aligned} \text{Then } \|z_m - z_n\| &= \|z_m - z_{m+1} + z_{m+1} - z_{m+2} + z_{m+2} - \dots - z_{n-1} + z_{n-1} - z_n\| \\ &\leq \|z_m - z_{m+1}\| + \|z_{m+1} - z_{m+2}\| + \dots + \|z_{n-1} - z_n\| \\ &< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \dots + \frac{1}{2^{n-1}} \end{aligned}$$

$$\Rightarrow \|z_m - z_n\| < \frac{1}{2^{m-1}}$$

$\Rightarrow \{z_n\}$ is Cauchy sequence in X .

Since X is complete nls.

$\therefore \exists z \in X$ such that $z_n \rightarrow z$ as $n \rightarrow \infty$

$$\text{or } x_n + M \rightarrow x + M$$

$$\Rightarrow \|z_n - z\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{or } \|(x_n + M) - (x + M)\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ on } X/M$$

Hence X/M is complete nls.

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★ Riesz Lemma :- Let Y be a proper closed linear subspace of a normed linear space X over the field K . Let $0 < \alpha < 1$. Then there exists some $x_\alpha \in X$ such that

$$\|x_\alpha\| = 1 \text{ and } \inf_{y \in Y} \|x_\alpha - y\| \geq \alpha$$

Proof :

Let Y be a proper closed linear subspace of a nls X on the field K .

So by the definition of proper subspace,

$$X - Y \neq \emptyset$$

$$\text{Let } x \in X - Y \text{ and } d = \inf_{y \in Y} \|x - y\|$$

Claim : $d > 0$

Verification : If $d = 0$

$$\begin{aligned} \text{then } \inf_{y \in Y} \|x - y\| &= 0 \\ \Rightarrow \exists \text{ a seq}^n \{y_k\} \subset Y \text{ s.t. } \lim_{k \rightarrow \infty} \|x - y_k\| &= 0 \\ \Rightarrow \lim_{k \rightarrow \infty} y_k &= x \end{aligned}$$

It follows that x is a limit point of $\{y_k\} \subset Y$.
 Since Y is closed,
 Therefore it contains all the limit points
 i.e. $x \in Y$.
 which contradicts the fact that $x \in X - Y$.
 Hence $d \neq 0$ and $d > 0$.

$$\begin{aligned} \text{Since given that } 0 < \alpha < 1 \\ \text{Therefore } \frac{1}{\alpha} > 1 &\Rightarrow \frac{d}{\alpha} > d \Rightarrow d < \frac{d}{\alpha} \\ \Rightarrow \inf_{y \in Y} \|x - y\| &< \frac{d}{\alpha} \\ \Rightarrow \exists \text{ a point } y_0 \in Y \text{ such that} \\ \|x - y_0\| &< \frac{d}{\alpha} \Rightarrow \frac{1}{\|x - y_0\|} > \frac{1}{d/\alpha} \quad \text{--- (i)} \end{aligned}$$

$$\text{Let } x_\alpha = \frac{x - y_0}{\|x - y_0\|}$$

$$\text{Then } \|x_\alpha\| = \frac{\|x - y_0\|}{\|x - y_0\|} \Rightarrow \|x_\alpha\| = 1$$

$$\text{and } \|x_\alpha - y\| = \left\| \frac{x - y_0}{\|x - y_0\|} - y \right\| = \frac{\|x - y_0 - y\| \|x - y_0\|}{\|x - y_0\|}$$

$$= \frac{\|x - [y_0 + y\|x - y_0\|]\|}{\|x - y_0\|}$$

$$= \frac{\|x - y_1\|}{\|x - y_0\|}$$

$$\text{where } y_1 = y_0 + y\|x - y_0\|$$

$$\geq \frac{d}{d/\alpha} = \alpha$$

[by (i)]

$$\Rightarrow \|x_\alpha - y\| \geq \alpha$$

$$\Rightarrow \inf_{y \in Y} \|x_\alpha - y\| \geq \alpha$$

17/2-5

Remark: • Every closed and bounded subset of a finite dimensional nls is compact.

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• Every finite dimensional nls is complete i.e. Banach space.

• Every finite dimensional nls is closed.

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Theorem: (51) Let X be a nls. The closed unit ball

$$B = \{x \in X : \|x\| \leq 1\}$$

in X is compact iff X is finite dimensional.

OR.

The closed unit ball in a normed linear space is compact iff the space is finite dimensional.

Proof:

Let X be a normed linear space

and let $B = \{x \in X : \|x\| \leq 1\}$ be a closed unit ball in X .

First suppose that X is finite dimensional

Then we have to show that B is compact.

Since B is closed and bounded.

Also we know that "Every closed and bounded subset of a finite dimensional nls is compact".

$\Rightarrow B$ is compact.

Conversely, suppose that B is compact.

and suppose that X is infinite dimensional.

i.e. $\dim X = \infty$

Choose $x_1 \in X$ with $\|x_1\| = 1$

Then the linear space X_1 of x_1 is one-dimensional subspace of X .

i.e. x_1 generates a one-dimensional subspace X_1 of X .

Since X_1 is subspace of X and X is nls.

$\Rightarrow X_1$ is also nls.

Also, "Every finite dimensional nls is closed".

$\Rightarrow X_1$ is closed.

Now X_1 is proper closed subspace of X and $\dim X = \infty$

Then by Riesz lemma, $\exists x_2 \in X$ such that

$$\|x_2\| = 1 \quad \text{and} \quad \|x_2 - x_1\| \geq \frac{1}{2}$$

Suppose X_2 is a subspace generated by the span $\{x_1, x_2\}$. and $\dim X_2 = 2$

$\Rightarrow X_2$ is a proper closed subspace of X .

Again, by the Riesz lemma, $\exists x_3 \in X$ such that

$$\|x_3\| = 1 \quad \text{and} \quad \|x_3 - x_1\| \geq \frac{1}{2} \quad \& \quad \|x_3 - x_2\| \geq \frac{1}{2}$$

Continuing the process, we find a seqⁿ $\{x_n\} \subset X$ s.t.

$$\|x_n\| = 1 \quad \text{and} \quad \|x_n - x_m\| \geq \frac{1}{2} \quad \forall n > m \& n \neq m$$

Since $\|x_n\| = 1 \Rightarrow x_n \in B \quad \forall n$ [By defⁿ of B]

$\therefore \{x_n\}$ is a seqⁿ in B that is not convergent.

which contradicts the compactness of B .

So our assumption that X is infinite dimensional is wrong.

Hence X is finite dimensional.

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* Isometric or Linearly Homeomorphic or Topologically Isomorphic :-

Let X and Y are normed linear spaces. Then X and Y are said to be isometric or linearly homeomorphic or topologically isomorphic if there exists a linear map $F: X \rightarrow Y$ which is also a homeomorphism.

* Linearly Isometric or Isometrically Isomorphic :-

Two normed linear spaces X and Y are said to be linearly isometric or isometrically isomorphic if there exists a linear map $F: X \rightarrow Y$ which is also an isometry.

* Equivalent Normed Linear Spaces :- Let X and Y are two normed linear spaces. Then X and Y are said to be an equivalent normed linear spaces if they are isometrically isomorphic.

[18-19] 1.5 [20-21] 1.5 [22-23] 1.5

* Equivalent Norm :- Let X be a normed linear space. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X . Then the two norms are said to be equivalent on X if they induced equivalent metrics on X i.e. they induced the topology on X . This is denoted by $\|\cdot\|_1 \sim \|\cdot\|_2$.

OR,

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent

iff any open ball in $\|\cdot\|_1$ contains an open ball in $\|\cdot\|_2$ and viceversa.

OR.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called equivalent iff for a sequence $\{x_n\}$ in X , the convergence $x_n \rightarrow x$ in $\|\cdot\|_1$ is also convergence in $\|\cdot\|_2$ and viceversa.

Theorem: Let X be a normed linear space. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on X . Then these norms on X are equivalent iff $\exists$ positive constants m and M such that

$$m \cdot \|x\|_1 \leq \|x\|_2 \leq M \cdot \|x\|_1 \quad \forall x \in X.$$

$$\text{or } m \cdot \|x\|_2 \leq \|x\|_1 \leq M \cdot \|x\|_2 \quad \forall x \in X.$$

Proof:

Let X be a normed linear space.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms defined on X .

First suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent i.e. $\|\cdot\|_1 \sim \|\cdot\|_2$

Then we have to show that $\exists m$ & M s.t.

$$m \cdot \|x\|_1 \leq \|x\|_2 \leq M \cdot \|x\|_1$$

Let two spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ be denoted by X_1 and X_2 respectively.

Let $i_1: X_1 \rightarrow X_2$ and $i_2: X_2 \rightarrow X_1$ be two operators such that $i_1(x) = x$ and $i_2(x) = x$ i.e. i_1 and i_2 are identity operators.

Now, we show that i_1 and i_2 are continuous.

For this, let $\{x_n\} \in X$ be a seqⁿ converging to a point $x \in X$. i.e. $x_n \rightarrow x$ as $n \rightarrow \infty$

$$\text{Now } \|i_1(x_n) - i_1(x)\|_2 = \|x_n - x\|_1 \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{i.e. } \|i_1(x_n) - i_1(x)\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

$\Rightarrow i_1$ is continuous.

Similarly, i_2 is continuous.

Since i_1 and i_2 are continuous.

$\Rightarrow i_1$ and i_2 must be bounded.

Also, i_1 and i_2 are linear.

So $\exists$ positive constants $M > 0$ and $K > 0$ such that

$$\left. \begin{aligned} \|i_1(x)\|_2 &\leq M \cdot \|x\|_1 \\ \text{and } \|i_2(x)\|_1 &\leq K \cdot \|x\|_2 \end{aligned} \right\} \text{--- (i)}$$

$$\left. \begin{aligned} \text{By eqⁿ (i), } \|x\|_2 &\leq M \cdot \|x\|_1 \\ \text{and } \|x\|_1 &\leq K \cdot \|x\|_2 \Rightarrow \frac{1}{K} \|x\|_1 \leq \|x\|_2 \end{aligned} \right\} \text{--- (ii)}$$

$$\begin{aligned} \text{By eqⁿ (ii), } \frac{1}{K} \|x\|_1 &\leq \|x\|_2 \leq M \cdot \|x\|_1 & [\text{Let } \frac{1}{K} = m] \\ \Rightarrow m \cdot \|x\|_1 &\leq \|x\|_2 \leq M \cdot \|x\|_1 \end{aligned}$$

Conversely, suppose the condition

$$m \cdot \|x\|_1 \leq \|x\|_2 \leq M \cdot \|x\|_1 \text{ holds, } \forall m, M > 0.$$

Then we have to show that $\|\cdot\|_1 \sim \|\cdot\|_2$

For this, let $\{x_n\}$ be a seqⁿ in X such that $\|x_n - x\|_1 \rightarrow 0$ as $n \rightarrow \infty$

Now $\|x_n - x\|_2 \leq M \cdot \|x_n - x\|_1 \rightarrow 0$ as $n \rightarrow \infty$
 $\Rightarrow \|x_n - x\|_2 \rightarrow 0$ as $n \rightarrow \infty$

Similarly, let $\{x_n\}$ be a seqⁿ in X such that
 $\|x_n - x\|_2 \rightarrow 0$ as $n \rightarrow \infty$

So $n \cdot \|x_n - x\|_1 \leq \|x_n - x\|_2$

$\Rightarrow \|x_n - x\|_1 \leq \frac{1}{n} \cdot \|x_n - x\|_2 \rightarrow 0$ as $n \rightarrow \infty$

$\Rightarrow \|x_n - x\|_1 \rightarrow 0$ as $n \rightarrow \infty$

Hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent i.e. $\|\cdot\|_1 \sim \|\cdot\|_2$

Example: The norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent norms on K^n , where $K = \mathbb{R}$ or $\mathbb{C}$.

Solution: Let $x \in X = K^n$, where $K = \mathbb{R}$ or $\mathbb{C}$
 i.e. $x = \{x_1, x_2, \dots, x_n\} \in K^n$.

$$\text{Then } \|x\|_1 = \left[\sum_{i=1}^n |x_i|^1 \right]^{1/1} = \sum_{i=1}^n |x_i|$$

$$\|x\|_2 = \left[\sum_{i=1}^n |x_i|^2 \right]^{1/2}$$

$$\text{and } \|x\|_\infty = \max \{ |x_i| : 1 \leq i \leq n \}$$

$$\text{Now, } \|x\|_\infty = \max \{ |x_i| : 1 \leq i \leq n \}$$

$$\Rightarrow \|x\|_\infty \leq \sum_{i=1}^n |x_i| \leq n \cdot \max \{ |x_i| : 1 \leq i \leq n \}$$

$$\Rightarrow 1 \cdot \|x\|_\infty \leq \|x\|_1 \leq n \cdot \|x\|_\infty$$

$$\text{Thus } \|\cdot\|_\infty \sim \|\cdot\|_1$$

— (i)

$$\begin{aligned} \text{Now, } \|x\|_2 &= \left[\sum_{i=1}^n |x_i|^2 \right]^{1/2} \leq \sum_{i=1}^n |x_i| = \sum_{i=1}^n |x_{i-1}| \\ &\leq \left[\sum_{i=1}^n |x_i|^2 \right]^{1/2} \left[\sum_{i=1}^n 1^2 \right]^{1/2} \\ &\leq \|x\|_2 \cdot \sqrt{n} \end{aligned}$$

- Hamel Basis: A linearly independent set which is maximal w.r.to the set of inclusion.

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$$\Rightarrow 1 \cdot \|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \cdot \|x\|_2$$

$$\text{Thus } \|\cdot\|_2 \sim \|\cdot\|_1 \quad \text{--- (ii)}$$

$$\text{By eqn (i) and (ii), } \|\cdot\|_\infty \sim \|\cdot\|_2$$

Hence $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent.

1918 1928

Theorem: In a finite dimensional normed linear space or linear space all the norms are equivalent.

Proof:

Let X be a finite dimensional linear space.
i.e. $\dim X = n < \infty$

Let $\{e_1, e_2, \dots, e_n\}$ be a hamel basis of X .

Then any $x \in X$, so we can uniquely written as -

$$x = x_1 e_1 + x_2 e_2 + \dots + x_n e_n \quad \text{--- (i)}$$

$$\text{Let } \|x\| = \max \{|x_1|, |x_2|, \dots, |x_n|\} = \max_{1 \leq i \leq n} |x_i|. \quad \text{--- (ii)}$$

This is a norm $\|\cdot\|$ on X .

Suppose $\|\cdot\|_1$ be arbitrary norm on X .

Then by (i), $\|x\|_1 = \|x_1 e_1 + x_2 e_2 + \dots + x_n e_n\|_1$

$$\leq \|x_1 e_1\|_1 + \|x_2 e_2\|_1 + \dots + \|x_n e_n\|_1$$

$$\leq |x_1| \|e_1\|_1 + |x_2| \|e_2\|_1 + \dots + |x_n| \|e_n\|_1$$

$$\leq \max_{1 \leq i \leq n} |x_i| [\|e_1\|_1 + \|e_2\|_1 + \dots + \|e_n\|_1]$$

$$\Rightarrow \|x\|_1 \leq K_1 \cdot \|x\| \quad \text{--- (iii) [Let } K_1 = \|e_1\|_1 + \dots + \|e_n\|_1 \text{]}$$

Next, we show that $K_2 \cdot \|x\| \leq \|x\|_1$

For this, we use induction method on n .

- For $n=1$

$$\|x\|_1 = \|x+e_1\|_1 = |x_1| \cdot \|e_1\|_1 = \max_{i=1} |x_i| \cdot K_2$$

$$\Rightarrow \|x\|_1 = K_2 \cdot \|x\| \quad \text{--- (iv) [Let } K_2 = \|e_1\|_1 \text{ by (iii)]}$$

By (iii), (iv) $K_2 \cdot \|x\| = \|x\|_1 \leq K_1 \cdot \|x\|$

- For $n = n-1$

Suppose that the inequality holds for $n-1$ and $n > 1$

Now, let M be the subspace of X , spanned by $\{e_1, e_2, \dots, e_{n-1}\}$

Then $\dim M = n-1$

Thus by our assumption, $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent i.e. $\|\cdot\| \sim \|\cdot\|_1$ on M .

Since every finite dimensional nls is complete.

$\Rightarrow M$ is complete w.r. to $\|\cdot\|$ and $\|\cdot\|_1$.

Also finite dimensional nls is closed.

$\Rightarrow M$ is closed.

Now if e_n be the n th basis.

Then consider the set, $e_n + M = \{e_n + z \mid z \in M\}$

$\times \left\{ \begin{array}{l} \text{Also, the map } z \rightarrow e_n + z \\ \text{i.e. the map } M \rightarrow e_n + M. \end{array} \right.$

Let $x, y \in M$

$$\text{Then } \|(e_n + x) - (e_n + y)\| = \|x - y\|$$

$\Rightarrow$ The mapping $z \rightarrow e_n + z$ is isometric. ?

$\Rightarrow$ The map $M \rightarrow e_n + M$ is isometric.

Since M is closed

$\therefore e_n + M$ is also closed

$\Rightarrow (e_n + M)^c$ is open.

Claim : $0 \notin e_n + M$

Verification : Suppose on contrary $0 \in e_n + M$

i.e. $0 = e_n + x_1 e_1 + x_2 e_2 + \dots + x_{n-1} e_{n-1}$

But this expression is not possible,
because $\{e_1, e_2, \dots, e_n\}$ be linearly independent.

$\therefore 0 \notin e_n + M$

$\Rightarrow 0 \in (e_n + M)^c$

So there must be a constant $c_n > 0$ such that the
open ball $B(0, c_n) = \{x : \|x\|_1 < c_n\}$

is completely contained in $(e_n + M)^c$

This implies that, if any $x \in e_n + M$, then $\|x\|_1 \geq c_n$

$\Rightarrow \|e_n + x_1 e_1 + \dots + x_{n-1} e_{n-1}\|_1 \geq c_n$ — (v)

If $x_n \neq 0$,

then we consider $x_i = \frac{x_i}{x_n} \quad \forall i = 1, 2, \dots, n-1$

$\Rightarrow \|e_n + \frac{x_1}{x_n} e_1 + \frac{x_2}{x_n} e_2 + \dots + \frac{x_{n-1}}{x_n} e_{n-1}\|_1 \geq c_n$

$\Rightarrow \|x_n e_n + x_1 e_1 + x_2 e_2 + \dots + x_{n-1} e_{n-1}\|_1 \geq c_n \cdot |x_n|$

$\Rightarrow \|x\|_1 \geq c_n \cdot |x_n|$

[by (i)]

$\Rightarrow \|x\|_1 \geq \min_{1 \leq i \leq n} c_i \cdot \max_{1 \leq i \leq n} |x_i|$

[Let $K_3 = \min_{1 \leq i \leq n} c_i$]

$\Rightarrow \|x\|_1 \geq K_3 \cdot \|x\|_\infty$

— (vi)

[by (ii)]

By (iii) & (vi), $K_3 \cdot \|x\| \leq \|x\|_1 \leq K_1 \cdot \|x\|$

Thus $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent

i.e. $\|\cdot\| \sim \|\cdot\|_1$

Since $\|\cdot\|$ and $\|\cdot\|_1$ are arbitrary.

Hence all the norms are equivalent on a finite dimensional linear space or nls.

Ex 2 * Sequentially Compact :- A subset E of a nls X is said to be sequentially compact if every infinite sequence of elements of E has a sub-sequence, which converges to an element of E .

* Relatively Sequentially Compact :- A subset E of a nls X is said to be relatively sequentially compact if every sequence in E has a sub-sequence, which converges to an element of X .

Note :- • A subset of a nls which is closed and relatively compact is bounded.

- The closure of any relatively compact set in a nls is compact.

* Banach Space :- A Banach space X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a norm $\|\cdot\|$ which is complete with respect to this norm, i.e. every Cauchy sequence in X is convergent.

Examples:

- $X = \mathbb{R}^n$, then $\mathbb{R}^n$ is a vector space over a field $\mathbb{R}$.

Define $\|\cdot\|: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} \quad ; \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

$\therefore (\|\cdot\|, \mathbb{R}^n)$ is a Banach space.

- l_p -space, $p \geq 1$ is the space of numerical seqn $\{x_n\}$ for which

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

with the norm $\|x\| = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p}$

is a Banach space.

21-22/05

22-23/05

Theorem: In a normed linear space every convergent sequence is a Cauchy sequence.

Proof:

Suppose that X is a normed linear space and let $\{x_n\}$ be a convergent seqn in X .

Then we have to show that $\{x_n\}$ is a Cauchy sequence in X .

Let $\{x_n\}$ converge to $x_0 \in X$, then $\exists$ a positive integer n_0 such that

$$\|x_n - x_0\| < \frac{\epsilon}{2} \quad \forall n \geq n_0 \quad \text{--- (i)}$$

In particular $n=m$,

$$\|x_m - x_0\| < \frac{\epsilon}{2} \quad \forall m \geq n_0 \quad \text{--- (ii)}$$

So we have,

$$\begin{aligned}\|x_n - x_m\| &= \|x_n - x_0 + x_0 - x_m\| \\ &\leq \|x_n - x_0\| + \|x_m - x_0\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\end{aligned}$$

$$\Rightarrow \|x_n - x_m\| < \varepsilon \quad \forall n, m \geq n_0$$

This shows that the seqⁿ $\{x_n\}$ is Cauchy sequence.

18-19/4

Theorem: Every complete subspace of a normed linear space is closed.

Proof:

Let X be a normed linear space and M be a complete subspace of X .

Then we have to show that M is closed.

We know that 'a set is said to be closed if it contains all its limit points'.

Let x_0 be any limit point of M .

Then every positive integer n , the open sphere

$$S(x_0, \frac{1}{n}) = \{x : \|x - x_0\| < \frac{1}{n}\}$$

must contain a point x_n of M .

Suppose $\{x_n\}$ be a seqⁿ in M which converges to x_0 .

$\Rightarrow \{x_n\}$ is a Cauchy seqⁿ in M .

[$\because$ In a nls every convergent seqⁿ is a Cauchy seqⁿ]

Also, M is complete.

$\Rightarrow x_0 \in M$.

Since x_0 is arbitrary.

Therefore M contains all its limit points.

$\Rightarrow M$ is closed.

19-20/4

Theorem: Every finite dimensional normed linear space is complete or Banach space.

Proof:

Let X be a finite dimensional normed linear space and $\dim X = n$.

Suppose $B = \{e_1, e_2, \dots, e_n\}$ be a basis for X .

So, for $x \in X$

$$x = \sum_{i=1}^n \alpha_i e_i$$

where $\alpha_i ; i=1, 2, \dots, n$ are scalars.

Since all the norms on a finite dimensional nls are equivalent.

$$\text{Let } \|x\|_0 = \max_i |\alpha_i| \quad \text{--- (i)}$$

Suppose $\{x_n\}$ be a Cauchy seqⁿ in X .

$$\therefore x_n = \alpha_1^n e_1 + \alpha_2^n e_2 + \dots + \alpha_n^n e_n$$

$$\Rightarrow x_n = \sum_{i=1}^n \alpha_i^n e_i$$

In particular,

$$x_k = \sum_{i=1}^n \alpha_i^k e_i$$

such that all n scalars are unique.

$$\begin{aligned} \text{Now, } x_n - x_m &= \sum_{i=1}^n \alpha_i^n e_i - \sum_{i=1}^n \alpha_i^m e_i \\ &= \sum_{i=1}^n |\alpha_i^n - \alpha_i^m| e_i \end{aligned}$$

$$\therefore \|x_n - x_m\|_0 = \max_i |x_i^n - x_i^m| \quad \text{--- (ii) [by (i)]}$$

Since $\{x_n\}$ is a Cauchy sequence,

$$\therefore \|x_n - x_m\|_0 \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow |x_i^n - x_i^m| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{[by (ii)]}$$

Since all x_i 's belongs to $\mathbb{R}$ or $\mathbb{C}$ and each of is complete.

So, $\exists$ scalar $x_1, x_2, \dots, x_n$ such that

$$x_i^n \rightarrow x \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow |x_i^n - x| = 0$$

$$\Rightarrow \max_i |x_i^n - x| = 0$$

$$\Rightarrow \|x_n - x\|_0 = 0$$

$$\Rightarrow x_n \rightarrow x \quad \text{as } n \rightarrow \infty \quad \forall x \in X.$$

Hence X is complete.

2017/205

Theorem: In a finite dimensional normed linear space X , any subset M of X is compact iff M is closed and bounded.

Proof:

Let X be a finite dimensional normed linear space

and let M is compact subset of X .

Then we have to show that M is bounded and closed.

Let x is limit point of M .

$\Rightarrow$ There exists a sequence $\{x_n\}$ in M such that
$$x_n \rightarrow x$$

$\Rightarrow x \in M$

[$\because M$ is compact]

$\Rightarrow M$ contains all of its limit points.

$\Rightarrow M$ is closed.

Suppose if possible M is unbounded.

Then M contains an unbounded sequence $\{y_n\}$ such that
$$d(x, y_n) > n \quad \text{for fixed } x \in X.$$

$\Rightarrow M$ does not have a convergent subsequence of $\{y_n\}$.

$\Rightarrow M$ is not compact
which is contradiction.

So, M is bounded.

Conversely, let M is closed and bounded.

Then we have to show that M is compact.

Since X is finite dimensional normed linear space

and let $\dim X = n$

Suppose $B = \{e_1, e_2, \dots, e_n\}$ be a basis for X .
and let $\{x_m\}$ be any sequence in M

Then $x_m = \alpha_1^m e_1 + \alpha_2^m e_2 + \dots + \alpha_n^m e_n$ — (1)
where α_i^m ; $i = 1, 2, \dots, n$ are scalars.

Since M is bounded.

$\Rightarrow \exists$ a positive integer k such that

$$\|x_m\| \leq k \quad \forall m$$

$$\Rightarrow \|x_m\| = \left\| \sum_{i=1}^n \alpha_i^m e_i \right\| \leq k$$

$$\Rightarrow k \geq \left\| \sum_{i=1}^n \alpha_i^m e_i \right\|$$

$$\Rightarrow k \geq c \sum_{i=1}^n |\alpha_i^m|, \quad c > 0 \quad [\text{by lemma}]$$

$$\Rightarrow |\alpha_i^m| \leq k/c = K (\text{say}) \quad \forall m, i=1, 2, \dots, n$$

$$\Rightarrow |\alpha_i^m| \leq K \quad \forall m, \forall i=1, 2, \dots, n$$

$\Rightarrow \{\alpha_i^m\}$ is bounded sequence of scalars.

[Since scalars are $\mathbb{R}$ or $\mathbb{C}$ and every bounded sequence in $\mathbb{R}$ or $\mathbb{C}$ has convergent.]

So by Bolzano Weierstrass Theorem, "Every bounded sequence of real numbers has a convergent sub-sequence".

$\Rightarrow \{\alpha_i^m\}$ has a convergent subsequence $\{\alpha_i^{m_j}\}$, which converges to α .

Since $\alpha_i^m \rightarrow \alpha$

$$\Rightarrow x_m \rightarrow x$$

$\Rightarrow \{x_m\}$ has a convergent sub-sequence $\{y_{n,m}\}$ which converges to x .

$$\text{i.e. } \{y_{n,m}\} \rightarrow x \in M$$

such that

$$\text{for } x_m = \alpha_1^m e_1 + \alpha_2^m e_2 + \dots + \alpha_n^m e_n$$

we have

$$y_{1,m} = x_1^m e_1 + x_2^m e_2 + \dots + x_n^m e_n$$

$$y_{2,m} = x_1^m e_1 + x_2^m e_2 + \dots + x_n^m e_n$$

...

$$y_{n,m} = x_1^m e_1 + x_2^m e_2 + \dots + x_n^m e_n$$

Clearly by definition of compactness,
 $\Rightarrow M$ is compact.

This completes the proof.

[17/1.5] [19/1.5] [20/1.5] [22/1.5]

10/08/23
Bhavika Gupta

(Q7) * Bounded Linear Transformation :- Let K denotes the real field R or the complex field C and let X and Y be normed linear spaces over K .

A linear transformation $T: X \rightarrow Y$ is said to be a bounded linear transformation if $\exists$ a constant $M > 0$ such that $\|Tx\| \leq M \cdot \|x\| \quad \forall x \in X$.

Note: • If T is a bounded linear transformation, then the norm of T is defined by

$$\rightarrow \|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

$$\rightarrow \|T\| = \inf \{ M : M > 0 ; \|Tx\| \leq M \cdot \|x\| \} \quad \forall x \in X$$

$$\rightarrow \|T\| = \sup \{ \|Tx\| : x \in X ; \|x\| = 1 \}$$

* Bounded Linear Operator :- If T is a bounded linear transformation from a nls X into itself i.e. $T: X \rightarrow X$. Then T is called a bounded linear operator.

[21/1.5]

* Bounded Linear Functional :- If T is a bounded linear transformation from a nls X into a field K i.e. $T: X \rightarrow K$. Then T is called a bounded linear functional.

(62)

Examples:

- The identity operator $I: X \rightarrow X$ on a nls $X \neq \{0\}$ is a bounded linear operator with the norm

$$\|I\| = 1$$

$$\text{i.e. } \|I\| = \sup \left\{ \frac{\|Ix\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

$$= \sup \left\{ \frac{\|x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

$$\Rightarrow \|I\| = 1$$

- The zero transformation $0: X \rightarrow Y$ on a nls X is a bounded linear transformation and has the norm $\|0\| = 0$

$$\text{i.e. } \|0\| = \sup \left\{ \frac{\|0x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} = \sup \{0\}$$

$$\Rightarrow \|0\| = 0$$

- The norm $\|\cdot\|: X \rightarrow \mathbb{R}$ on a nls X is not a linear functional also not a bounded linear functional. But it is a sublinear functional.

Theorem: For a bounded linear transformation T the following conditions are equivalent:

$$(1.) \|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

$$(2.) \|T\| = \inf \{ M : M > 0 ; \|Tx\| \leq M \cdot \|x\| \}$$

$$(3.) \|T\| = \sup \{ \|Tx\| : x \in X ; \|x\| \leq 1 \}$$

$$(4.) \|T\| = \sup \{ \|Tx\| : x \in X ; \|x\| = 1 \}$$

Proof:

Given that T is a bounded linear transformation with the norm

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

$$\Rightarrow \|T\| \geq \frac{\|Tx\|}{\|x\|}$$

— (i)

$$\Rightarrow \|x\| \cdot \|T\| \geq \|Tx\|$$

—(ii)

• For (1.) $\Rightarrow$ (2.)

Since T is a bounded linear transformation.

So by defⁿ, $\exists$ a constant $M \geq 0$ such that

$$\|Tx\| \leq M \cdot \|x\| \quad \forall x \in X, \|x\| \neq 0$$

$$\Rightarrow \frac{\|Tx\|}{\|x\|} \leq M$$

$$\forall x \in X, \|x\| \neq 0 \quad \text{—(iii)}$$

Taking supremum,

$$\Rightarrow \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X, \|x\| \neq 0 \right\} \leq M$$

$$\Rightarrow \|T\| \leq M$$

This relation is true for all $x \in X$ and $M \geq 0$

So, taking infimum in a R.H.S.

$$\Rightarrow \|T\| \leq \inf \{ M : M \geq 0, \|Tx\| \leq M \cdot \|x\| \}$$

—(iv)

By eqn (i) and (iii),

$$\|T\| \geq \frac{\|Tx\|}{\|x\|} = M$$

$$\Rightarrow \|T\| \geq \inf \{ M : M \geq 0, \|Tx\| \leq M \cdot \|x\| \}$$

—(v)

By eqn (iv) and (v),

$$\|T\| = \inf \{ M : M \geq 0, \|Tx\| \leq M \cdot \|x\| \}$$

• For (1.) $\Rightarrow$ (3.)

If $\|x\| \leq 1$, then by eqn (ii),

$$\|T\| \geq \|Tx\|$$

$$\Rightarrow \|T\| \geq \sup \{ \|Tx\| : x \in X, \|x\| \leq 1 \}$$

—(vi)

By the property of supremum, $\exists \varepsilon > 0$

$$\frac{\|Tx\|}{\|x\|} + \varepsilon \geq \|T\|$$

$$\Rightarrow \|Tx\| \geq \frac{\|Tx\|}{\|x\|} \geq \|T\| - \varepsilon$$

; if $\|x\| \leq 1$

$$\Rightarrow \sup \{ \|Tx\| : x \in X, \|x\| \leq 1 \} \geq \|T\| - \varepsilon$$

since ε is arbitrary, $\varepsilon \rightarrow 0$, we get

$$\Rightarrow \sup \{ \|Tx\| : x \in X, \|x\| \leq 1 \} \geq \|T\| \quad \text{--- (vii)}$$

By eqn (vi) and (vii),

$$\|T\| = \sup \{ \|Tx\| : x \in X, \|x\| \leq 1 \}$$

• For (1.) $\Rightarrow$ (4.)

If $\|x\| = 1$, then by eqn (ii),

$$\|T\| \geq \|Tx\|$$

$$\Rightarrow \|T\| \geq \sup \{ \|Tx\| : x \in X, \|x\| = 1 \} \quad \text{--- (viii)}$$

By the property of supremum, $\exists \varepsilon > 0$

$$\frac{\|Tx\|}{\|x\|} + \varepsilon \geq \|T\|$$

$$\Rightarrow \|Tx\| \geq \frac{\|Tx\|}{\|x\|} \geq \|T\| - \varepsilon \quad ; \text{ if } \|x\| = 1$$

$$\Rightarrow \sup \{ \|Tx\| : x \in X, \|x\| = 1 \} \geq \|T\| - \varepsilon$$

since ε is arbitrary, so $\varepsilon \rightarrow 0$,

$$\Rightarrow \sup \{ \|Tx\| : x \in X, \|x\| = 1 \} \geq \|T\| \quad \text{--- (ix)}$$

By eqn (viii) and (ix),

$$\|T\| = \sup \{ \|Tx\| : x \in X, \|x\| = 1 \}$$

• For (4.) $\Rightarrow$ (1.)

By the definition of norm of T ,

$$\|T\| = \sup \{ \|Tx\| : x \in X, \|x\| = 1 \}$$

$$\Rightarrow \|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X, \|x\| = 1 \right\}$$

$$\Rightarrow \|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X, \|x\| \neq 0 \right\}$$

(62)

16/8

18/4

21/2.5

22/2.5

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Theorem: Let X and Y be two normed linear spaces and let $T: X \rightarrow Y$ be a linear transformation. Then the following conditions are equivalent -

- (1.) T is bounded.
- (2.) T is continuous
- (3.) T is continuous at one point.

Proof:

Let X and Y be two normed linear spaces, and let $T: X \rightarrow Y$ be a linear transformation.

- For (1.) $\Rightarrow$ (2.) [22/2.5]

Suppose T is bounded,

Then by definition, $\exists$ a constant $M > 0$ such that

$$\|Tx\| \leq M \cdot \|x\| \quad \forall x \in X$$

Let $x_1, x_2 \in X$ be arbitrary such that for a given $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\begin{aligned} \|x_1 - x_2\| < \delta &\Rightarrow \|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \quad [\because T \text{ is linear}] \\ &\leq M \cdot \|x_1 - x_2\| \\ &< M \cdot \delta \quad [\text{let } \delta = \frac{\varepsilon}{M}] \\ &= \varepsilon \end{aligned}$$

$$\therefore \|x_1 - x_2\| < \delta \Rightarrow \|Tx_1 - Tx_2\| < \varepsilon$$

$\Rightarrow T$ is uniformly continuous.

$\Rightarrow T$ is continuous.

- For (2.) $\Rightarrow$ (3.)

Suppose T is continuous.

$\therefore T$ is continuous at each point of X .

$\Rightarrow T$ is continuous at one point.

• For (3.) $\Rightarrow$ (1.)

Let T is continuous at one point $x_0 \in X$

Then for any $\varepsilon > 0$, $\exists \delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < \varepsilon \quad \forall x \in X$$

Now, for any $\varepsilon = 1$, $\exists$ a $\delta > 0$ such that

$$\|x - x_0\| < \delta \Rightarrow \|Tx - Tx_0\| < 1 \quad \forall x \in X \quad \text{--- (i)}$$

Next, we replace x by $x + x_0$ in eqn (i),

$$\|x + x_0 - x_0\| < \delta \Rightarrow \|T(x + x_0) - Tx_0\| < 1$$

$$\Rightarrow \|x\| < \delta \Rightarrow \|Tx + Tx_0 - Tx_0\| < 1 \quad [\because T \text{ is linear transformation}]$$

$$\Rightarrow \|x\| < \delta \Rightarrow \|Tx\| < 1 \quad \forall x \in X \quad \text{--- (ii)}$$

I. If $x = 0$

Then it is trivially true that T is bounded.

II. If $x \neq 0$

Then take $x = \frac{\delta}{2} \cdot \frac{x}{\|x\|}$ in eqn (ii),

$$\therefore \left\| \frac{\delta}{2} \frac{x}{\|x\|} \right\| = \left| \frac{\delta}{2} \right| \cdot \frac{\|x\|}{\|x\|} = \frac{\delta}{2} < \delta$$

$$\Rightarrow \left\| T \left(\frac{\delta}{2} \frac{x}{\|x\|} \right) \right\| < 1$$

$$\Rightarrow \left| \frac{\delta}{2} \right| \frac{1}{\|x\|} \cdot \|Tx\| < 1$$

$$\Rightarrow \|Tx\| < \frac{2}{\delta} \|x\|$$

[Take $M = \frac{2}{\delta}$]

$$\Rightarrow \|Tx\| < M \cdot \|x\|$$

$\Rightarrow T$ is bounded.

Hence the conditions (1.), (2.) and (3.) are equivalent.

19/4 17/4 21/4 13/8 15/8

Theorem ⁽⁶²⁾: If X and Y are normed linear spaces, then the set of all bounded linear transformation from X into Y i.e. $B(X, Y)$ is a normed linear space with pointwise linear operations and the norm is defined by

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

Moreover, $B(X, Y)$ is a Banach space if Y is a Banach space.

Proof:

Let X and Y are two normed linear spaces.

Firstly, we have to show that $B(X, Y)$ is a normed linear space.

For this, let $T \in B(X, Y)$

Then by defⁿ of bounded linear transformation,

$$\|Tx\| \leq M \cdot \|x\| \quad \forall x \in X$$

And the norm of T is defined by

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}$$

1. $\|T\| \geq 0$, it is trivially true.

2. Let $T_1, T_2 \in B(X, Y)$, then

$$\begin{aligned} \|T_1 + T_2\| &= \sup \left\{ \frac{\|(T_1 + T_2)x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} \\ &= \sup \left\{ \frac{\|T_1x + T_2x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} \\ &\leq \sup \left\{ \frac{\|T_1x\| + \|T_2x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} \\ &= \sup \left\{ \frac{\|T_1x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} + \sup \left\{ \frac{\|T_2x\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} \\ &= \|T_1\| + \|T_2\| \end{aligned}$$

$$\Rightarrow \|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$$

$$\begin{aligned}
 3. \quad \|\lambda T\| &= \sup \left\{ \frac{\|\lambda Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} \\
 &= \sup \left\{ \frac{|\lambda| \|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} \\
 &= |\lambda| \cdot \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\}
 \end{aligned}$$

$$\Rightarrow \|\lambda T\| = |\lambda| \cdot \|T\|$$

$$4. \quad \text{If } \|T\| = 0$$

$$\text{Then } \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in X ; \|x\| \neq 0 \right\} = 0$$

$$\Rightarrow \|Tx\| = 0$$

$$\Rightarrow Tx = 0 \quad \text{if } x \neq 0$$

$$\Rightarrow T = 0$$

Thus $B(X, Y)$ is a normed linear space.

Suppose Y is a Banach space.

Then we have to show that $B(X, Y)$ is Banach space.

For this, let $\{T_n\}$ be a Cauchy seqn in $B(X, Y)$.

$\therefore$ For any $\varepsilon > 0$, $\exists$ positive integer $n_0 > 0$ such that

$$\|T_n - T_m\| < \varepsilon \quad \forall n, m \geq n_0 \quad \text{--- (i)}$$

Now, for any fixed vector $x \in X$

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\|$$

$$\leq \|T_n - T_m\| \cdot \|x\| < \varepsilon \cdot \|x\| = \varepsilon_1 \quad \text{[by (i)]}$$

$$\Rightarrow \|T_n x - T_m x\| < \varepsilon_1$$

$\Rightarrow \{T_n x\}$ is a Cauchy seqn in Y .

Since Y is Banach space.

$$\therefore \exists Tx \in Y \text{ such that } \lim_{n \rightarrow \infty} T_n x = Tx \quad \text{--- (ii)}$$

Next, we have to show that T is bounded & linear.

- T is linear :

Let $x, y \in X$ and α, β are scalar.

$$\begin{aligned} \text{Then } T(\alpha x + \beta y) &= \lim_{n \rightarrow \infty} T_n(\alpha x + \beta y) && [\text{by (ii)}] \\ &= \lim_{n \rightarrow \infty} [\alpha T_n x + \beta T_n y] \\ &= \alpha \lim_{n \rightarrow \infty} T_n x + \beta \lim_{n \rightarrow \infty} T_n y \end{aligned}$$

$$\Rightarrow T(\alpha x + \beta y) = \alpha T x + \beta T y$$

$\Rightarrow T$ is linear.

- T is bounded :

$$\text{Now, } \|T_n\| = \|T_n - T_{n_0} + T_{n_0}\| \leq \|T_n - T_{n_0}\| + \|T_{n_0}\|$$

$$< \varepsilon + \|T_{n_0}\| \quad [\text{by (i)}]$$

$$\leq \varepsilon + M$$

$$[\because \|T_{n_0}\| \leq M]$$

$$\Rightarrow \|T_n\| \leq M_n$$

$$[\text{Let } \varepsilon + M = M_n]$$

$$\text{Let } M' = \max \{M_n\} = \max \{M_1, M_2, \dots, M_{n_0-1}, \varepsilon + M\}$$

$$\Rightarrow \|T_n\| \leq M' \quad \forall n$$

$\Rightarrow \{T_n\}$ is bounded.

$$\therefore \|T x\| = \left\| \lim_{n \rightarrow \infty} T_n x \right\| = \lim_{n \rightarrow \infty} \|T_n x\| \leq \lim_{n \rightarrow \infty} \|T_n\| \cdot \|x\|$$

$$\Rightarrow \|T x\| \leq M' \cdot \|x\|$$

$\Rightarrow T$ is bounded.

Finally, we have to show that $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$

For $n, m \geq n_0$ and $\|x\| = 1$, we have

$$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\| \cdot \|x\|$$

$$\Rightarrow \|T_n x - T_m x\| < \varepsilon \cdot 1 \quad [\text{by (i)}]$$

$$\Rightarrow \|T_n x - T_m x\| < \varepsilon$$

Let n be fixed and $m \rightarrow \infty$

$$\Rightarrow \lim_{m \rightarrow \infty} \|T_n x - T_m x\| = \|T_n x - T x\| < \varepsilon \quad \forall n \geq n_0$$

Again, if $\|x\| = 1$

Then $\|T_n x - T x\| \leq \|T_n - T\| \|x\| = \|T_n - T\| < \varepsilon$

$\Rightarrow \|T_n - T\| < \varepsilon$

$\Rightarrow \lim_{n \rightarrow \infty} T_n = T$

i.e. $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$

$\Rightarrow$ The seqⁿ $\{T_n\}$ converges to T in $B(X, Y)$

Note: • $B(X)$ is a Banach space if nls X is Banach space.

• $B(X) =$ Set of all bounded linear operators from nls X into itself.

• $B(X, Y) =$ Set of all bounded linear transformation from nls X to nls Y .

18/1.5 19/1.5 22/1.5

⑥3 * Conjugate or Dual Space of X :- Let X be a normed linear

space. Then the set of all bounded linear functional defined on X to $\mathbb{R}/\mathbb{C}$ is called a conjugate or Dual space of X . It is denoted by X^* .

i.e. $T: X \rightarrow \mathbb{R}/\mathbb{C}$; $T \in X^*$

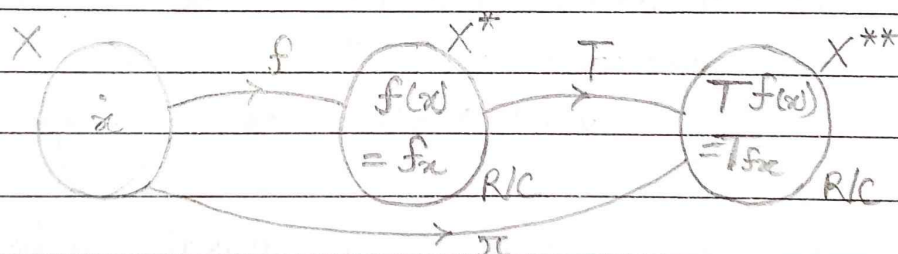
⑥4 * Second Conjugate or Second Dual Space of X :-

Let X be a normed linear space and X^* be a dual space of X . Then the dual of X^* i.e. the set of all bounded linear functional defined on X^*

is called a double conjugate or second conjugate or second dual space of X . It is denoted by X^{**} i.e. $F: X^* \rightarrow \mathbb{R}/\mathbb{C}$; $F \in X^{**}$

Note: • The conjugate space of X i.e. X^* is always a Banach space even if X is not complete.

⑥ ★. Reflexive Space :- A normed linear space is said to be a reflexive space if the natural mapping $\pi: X \rightarrow X^{**}$ or $J: X \rightarrow X^{**}$ is defined by $\pi(x)(f) = f(x) \quad \forall x \in X, f \in X^*$
or $\pi x(f) = f(x)$
or $\pi x: X^* \rightarrow X^{**}$ is onto.
i.e. $\pi(X) = X^{**}$.



Examples: • $\mathbb{R}^n$ space is always reflexive space.

i.e. $f: \mathbb{R}^n \rightarrow \mathbb{K}$, then $\mathbb{R}^n = \mathbb{R}^{n*} = \mathbb{R}^{n**}$

• The space l_p is reflexive for $1 < p < \infty$.

• The spaces c_0 , c , l_1 and l_∞ are not reflexive.

Note: • A closed subspace of reflexive Banach space is also reflexive.



- Let X be a reflexive normed linear space and suppose that $F \in X^{**}$. Then $\exists x \in X$ such that $f(x) = F(f) \quad \forall f \in X^*$

18/1.5+ 21/1.5*

* Strong Convergence of a Sequence :- Let $\{x_n\}$ be a sequence in a normed linear space X . Then the seqⁿ $\{x_n\}$ is said to be a strong convergence to x in X if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$

18/1.5+ 20/1.5 21/1.5*

* Weak Convergence of a Sequence :- Let $\{x_n\}$ be a sequence in a normed linear space X . Then the seqⁿ $\{x_n\}$ is said to be converges weakly to x in X if for every linear functional $f \in X^*$ $f(x_n) \rightarrow f(x)$ as $n \rightarrow \infty$

i.e. For every $\varepsilon > 0$, $\exists$ a natural number n_0 such that $|f(x_n) - f(x)| < \varepsilon \quad \forall n \geq n_0, f \in X^*$

In this case, we write $x_n \xrightarrow{w} x$ and x is called weak limit of the seqⁿ $\{x_n\}$.

Note :- Weak convergence is the convergence of seqⁿ of numbers.

* Weak* Convergence :- Let $\{f_n\}$ be a sequence of bounded linear functional on

a normed linear space X . Then the sequence $\{f_n\}$ is called weak* convergence or weakly* convergence to $f \in X^*$ such that

$$f_n(x) \rightarrow f(x)$$

In this case, we write $f_n \xrightarrow{w^*} f$

Note: • Every convergence sequence in nls X is weakly convergent but converse may not be true.

This notations are equivalent if X is a finite dimensional space.

- Weak convergence implies the weakly convergence but the converse is not true in general.

This notation $[w \text{ \& } w^*]$ be equivalent if the space X is reflexive space.

- The limit of weak convergence requires the uniqueness.
- If $\{x_n\}$ and $\{y_n\}$ converges weakly to x and y respectively i.e. $x_n \xrightarrow{w} x$ & $y_n \xrightarrow{w} y$
Then $x_n + y_n \xrightarrow{w} x + y$
- If $\{x_n\}$ converges weakly to x and a sequence of scalars $\{\alpha_n\}$ converges to α i.e. $\alpha_n \xrightarrow{w} \alpha$
Then $\alpha_n x_n \xrightarrow{w} \alpha x$

Theorem: Weak convergence :- Let $\{x_n\}$ be a weakly convergent sequence in a normed linear space X i.e. $x_n \xrightarrow{w} x$. Then

(1) The weak limit of $\{x_n\}$ is unique.

(2) Every subsequence of $\{x_n\}$ converges weakly to x .

(3) The sequence $\{\|x_n\|\}$ is bounded.

Proof:

Let $\{x_n\}$ be a weakly convergent sequence in a normed linear space X .

(1) We have to show that the weak limit of $\{x_n\}$ is unique.

For this, suppose $\{x_n\}$ converges weakly to x & y i.e. $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$

So by defⁿ, $\exists f \in X^*$ such that
 $f(x_n) \rightarrow f(x)$ and $f(x_n) \rightarrow f(y)$

Since $\{f(x_n)\}$ is a sequence of numbers.

$\therefore$ It's limits are unique.

$$\Rightarrow f(x) = f(y) \Rightarrow f(x) - f(y) = 0$$

$$\Rightarrow f(x-y) = 0 \Rightarrow x-y = 0$$

$$\Rightarrow \boxed{x = y}$$

(2) Since $\{f(x_n)\}$ is convergent sequence of numbers.

$\Rightarrow$ Every sub-sequence of $\{f(x_n)\}$ converges and has same limit as a seqⁿ $\{f(x_n)\}$.

Hence every sub-sequence of $\{x_n\}$ converges weakly to x .

(3.) Since $\{f(x_n)\}$ is convergent sequence of numbers.
 $\therefore$ It is bounded.

$\Rightarrow \exists$ a constant C_f depending on f but not depending on $n \in \mathbb{N}$ such that

$$|f(x_n)| \leq C_f \quad \forall n \in \mathbb{N}$$

Now, using a natural mapping (canonical mapping)
 $\pi: X \rightarrow X^{**}$ defined by

$$\pi(x) = F_x \quad \forall F_x \in X^{**}$$

$$\text{i.e. } \pi(x_n) = F_{x_n} \quad \forall F_{x_n} \in X^{**}$$

We can define $F_{x_n} \in X^{**}$ by

$$F_{x_n}(f) = f(x_n) \quad \forall f \in X^*$$

$$\Rightarrow |F_{x_n}(f)| = |f(x_n)| \leq C_f \quad \forall n \in \mathbb{N}$$

$\Rightarrow \{F_{x_n}(f)\}$ is bounded $\forall x \in X, f \in X^*, F_{x_n} \in X^{**}$

— (i)

Since X^* is complete nls.

[17/9-5]

Then by "Uniform Boundedness Theorem", "For every fixed x in a nls X , the functional F_x defined by $F_x(f) = f(x)$, $f \in X^*$ is bounded linear functional on X^* . So that $F_x \in X^{**}$ and hence the norm $\|F_x\| = \|x\|$."

So by using this result, we have

$$\|F_{x_n}\| = \|x_n\| \quad \text{— (ii)}$$

From eqn (i) and (ii), we get

$\{\|x_n\|\}$ is bounded.

This completes the proof.

Ques. Prove that in a finite dimensional n/s strong and weak convergence coincides. [21/2.5] [12/8]

Theorem: ^{Q2} Strong and Weak Convergence: Let $\{x_n\}$ be a sequence in a normed

linear space X . Then

(1.) Strong convergence implies weak convergence with the same limit.

(2.) Weak convergence does not necessarily implies strong convergence.

OR the converse of (1.) is not generally true.

[20/2.5] [19/2.5] (3.) If $\dim X < \infty$, then weak convergence implies strong convergence. [20/4] [14/8]

Proof:

(1.) Suppose $\{x_n\}$ converges strongly to x in a n/s X .
Then by defⁿ, $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$

Now for every $f \in X^*$

$$|f(x_n) - f(x)| = |f(x_n - x)| \quad [\because f \text{ is linear}]$$
$$\leq \|f\| \|x_n - x\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow f(x_n) \rightarrow f(x) \quad \text{as } n \rightarrow \infty$$

$$\Rightarrow x_n \xrightarrow{w} x \text{ in } X$$

Hence strong convergence implies weak convergence with the same limit.

(2.) Let $X = l_p$ for $1 < p < \infty$

Consider the seqⁿ $\{e_n\} = \{x_n\} \subset X$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\text{Then } \|e_n - 0\|_p = \|e_n\|_p = 1 \quad \forall n \in \mathbb{N}$$

$$\therefore e_n \not\rightarrow 0$$

Now we have to show that $e_n \xrightarrow{w} 0$

For this, consider the functional $f = \{f_i\} \subset l_q = X^*$

We take $f(x_n) = f_n$

Since $f \in L_q \Rightarrow \left[\sum_{n=1}^{\infty} |f_n|^q \right]^{1/q} < \infty$ or $\left[\sum_{n=1}^{\infty} |f_n - 0|^q \right]^{1/q} < \infty$

$\therefore$ By Cauchy's general principle / Cauchy's criteria,
 $[|f_n|^q]^{1/q} \rightarrow 0$ as $n \rightarrow \infty$

$$\Rightarrow |f_n| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow f_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus $e_n \xrightarrow{w} 0$ as $n \rightarrow \infty$ but $e_n \not\rightarrow 0$ as $n \rightarrow \infty$.

(3.) Suppose $x_n \xrightarrow{w} x$

and let X is a finite dimensional nls, $\dim X = k$

Then $\exists$ a basis $B = \{e_1, e_2, \dots, e_k\}$ for X .

$$\left. \begin{array}{l} \text{Take } x_n = \alpha_1^n e_1 + \alpha_2^n e_2 + \dots + \alpha_k^n e_k \quad \text{for } x_n \in X \\ \text{and } x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_k e_k \quad \text{for } x \in X \end{array} \right\} - (i)$$

Since $x_n \xrightarrow{w} x$

$$\Rightarrow f(x_n) \rightarrow f(x) \quad \forall f \in X^*$$

In particular, $f_1, f_2, \dots, f_k$ defined by

$$f_i(e_j) = \begin{cases} 1 & , \text{ if } i=j \\ 0 & , \text{ if } i \neq j \end{cases} \quad \text{for } i, j = 1, 2, \dots, k$$

Since $f_i(x_n) \in X^*$

$$\therefore f_i(x_n) = \alpha_1^n f_i(e_1) + \alpha_2^n f_i(e_2) + \dots + \alpha_k^n f_i(e_k)$$

$$\text{and } f_i(x) = \alpha_1 f_i(e_1) + \alpha_2 f_i(e_2) + \dots + \alpha_k f_i(e_k)$$

$$\text{Then } f_i(x_n) = \alpha_i^n \quad \text{and} \quad f_i(x) = \alpha_i$$

Therefore the set $\{f_1, f_2, \dots, f_k\}$ is a dual basis of $\{e_1, e_2, \dots, e_k\}$. ^{conjugate}

Now by weak convergence of $\{x_n\}$,

$$f_i(x_n) \rightarrow f_i(x) \quad \text{as } n \rightarrow \infty$$

$$\text{i.e. } \alpha_i^n \rightarrow \alpha_i \quad \text{as } n \rightarrow \infty \quad \forall i=1, 2, \dots, k$$

Then for any $\varepsilon > 0$, $\exists$ a positive integer n_0 s.t.

$$|\alpha_i^n - \alpha_i| < \varepsilon/M \quad \forall n \geq n_0 \quad ; \quad i=1, 2, \dots, k \quad \text{--- (ii)}$$

where $M = \sum_{i=1}^k \|e_i\|$. --- (iii)

$$\therefore \|x_n - x\| = \left\| \sum_{i=1}^k \alpha_i^n e_i - \sum_{i=1}^k \alpha_i e_i \right\| \quad [\text{by (i)}]$$

$$= \left\| \sum_{i=1}^k (\alpha_i^n - \alpha_i) e_i \right\|$$

$$\leq \sum_{i=1}^k |\alpha_i^n - \alpha_i| \|e_i\| < \frac{\varepsilon}{M} \times M \quad [\text{by (ii) \& (iii)}]$$

$$\Rightarrow \|x_n - x\| < \varepsilon \quad \forall n \geq n_0$$

$\Rightarrow x_n \rightarrow x$ i.e. $\{x_n\}$ converges strongly to x .

***. Total Subset :-** Let X be a normed linear space, then MCX is called total subset if linear span of M is dense in X .

$$\text{i.e. } \overline{\text{span } M} \subset X$$

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Lemma: Weak Convergence :- In a normed linear space X ,

$$x_n \xrightarrow{w} x \text{ iff}$$

(1.) The sequence $\{\|x_n\|\}$ is bounded.

(2.) For every element f of a total subset $M \subset X^*$
 $f(x_n) \rightarrow f(x)$

OR.

In the space l_p , where $1 \leq p < \infty$, $x_n \xrightarrow{w} x$ iff

(1.) The seqⁿ $\{ \|x_n\| \}$ is bounded.

(2.) For every fixed i , $\xi_i^{(n)} \rightarrow \xi_i$ as $n \rightarrow \infty$
here $\xi_i^{(n)} = f(x_n)$ and $\xi_i = f(x)$.

Proof:

Let X be a normed linear space and $x_n \xrightarrow{w} x$

(1.) Same as weak convergence (3.).

(2.) Since $x_n \xrightarrow{w} x$

So by definition of weak convergence, we have

$$f(x_n) \rightarrow f(x) \quad \forall f \in X^*$$

$$\text{i.e.} \quad f(x_n) \rightarrow f(x) \quad \forall x \in M \subset X^*$$

Conversely, suppose that the given condition holds.

Then we have to show that $x_n \xrightarrow{w} x$ in X .

Since $\{ \|x_n\| \}$ is bounded

$\Rightarrow \exists$ a constant c such that

$$\|x_n\| \leq c \quad \forall n \in \mathbb{N} \quad \text{--- (i)}$$

Since M is total subset of X^*

$\therefore$ For every $f \in X^*$, $\exists$ a seqⁿ $\{ f_i \} \subset \text{span of } M$
such that $f_i \rightarrow f$.

So, for any $\varepsilon > 0$, $\exists$ a positive integer j such that

$$|f_i - f| < \frac{\varepsilon}{3c} \quad \forall i \geq j \quad \text{--- (ii)}$$

Moreover, since $f_i \in \text{span of } M$.

So by condition (2.), $\exists$ a positive integer n_0 such that

$$|f_i(x_n) - f_i(x)| < \frac{\varepsilon}{3} \quad \forall n \geq n_0 \quad \text{--- (iii)}$$

$$\begin{aligned}
 \text{Now, } |f(x_n) - f(x)| &= |f(x_n) - f_i(x_n) + f_i(x_n) - f_i(x) + f_i(x) - f(x)| \\
 &\leq |f(x_n) - f_i(x_n)| + |f_i(x_n) - f_i(x)| + |f_i(x) - f(x)| \\
 &< |(f - f_i)x_n| + \frac{\epsilon}{3} + |(f_i - f)x| \quad (\text{by (iii)}) \\
 &= \|f - f_i\| \|x_n\| + \frac{\epsilon}{3} + \|f_i - f\| \|x\| \\
 &< \frac{\epsilon}{3c} \cdot c + \frac{\epsilon}{3} + \frac{\epsilon}{3c} \cdot c \quad (\text{by (i), (ii)})
 \end{aligned}$$

$$\Rightarrow |f(x_n) - f(x)| < \epsilon \quad \forall n \geq n_0$$

$$\Rightarrow f(x_n) \rightarrow f(x)$$

$$\Rightarrow x_n \xrightarrow{w_3} x \text{ in } X.$$

Space	Representation	Norm	Dual
C	$a \lim x_n + \sum a_n x_n$	$ a + \sum a_n $	ℓ_1
C_0	$\sum a_n x_n$	$\sum a_n $	ℓ_1
ℓ_p ($0 < p < \infty$)	$\sum a_n x_n$	$(\sum a_n ^p)^{1/p}$	ℓ_q
ℓ_p ($0 < p \leq 1$)	$\sum a_n x_n$	$\sup_n a_n $	ℓ_∞
ℓ_∞	$\int x_k \, d\mu$	$\int d\mu $	$ba(N)$
$C[0,1]$	$\int_0^1 x(t) \, dg(t)$	$\int_0^1 dg(t) $	$BV[0,1]$

where,

$ba(N)$ = Bounded finitely additive set function defined on set of all positive integers (Natural numbers).

$BV[0,1]$ = Set of all function of bounded variation which is define on $[0,1]$.

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Theorem: Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then l_p^* is isometrically isomorphic to l_q .

Proof:

Let $f \in l_p^*$

Since $\{e_1, e_2, \dots, e_n, \dots\}$ be a basis of l_p and $x \in l_p$

$\therefore$ We can express x by

$$x = \sum_k x_k e_k \quad \text{where } x = \{x_k\} \quad \text{--- (i)}$$

Define $\alpha_k = f(e_k)$

So by linearity and continuity of f , we have

$$f(x) = \sum_k x_k f(e_k) = \sum_k x_k \alpha_k \quad \text{--- (ii)}$$

For a given positive integer n ,

Choose x_k as follows,

$$x_k = \begin{cases} |\alpha_k|^{q-1} \operatorname{sgn} \bar{\alpha}_k & ; 1 \leq k \leq n \\ 0 & ; k > n \end{cases} \quad \text{--- (iii)}$$

where

$$\operatorname{sgn} c = \begin{cases} 0 & ; \text{if } c = 0 \\ \frac{c}{|c|} & ; \text{if } c \neq 0 \end{cases} \quad \text{--- (iv)}$$

and $\bar{\alpha}_k$ denotes the complex conjugate of α_k .

Now if $1 \leq k \leq n$

$$\text{Then } \alpha_k x_k = \alpha_k |\alpha_k|^{q-1} \operatorname{sgn} \bar{\alpha}_k$$

[by (iii)]

$$= \alpha_k |\alpha_k|^{q-1} \cdot \frac{\bar{\alpha}_k}{|\alpha_k|}$$

[by (iv)]

$$= |\alpha_k|^{q-1} \cdot \frac{|\alpha_k|^2}{|\alpha_k|}$$

$$[\because z \cdot \bar{z} = |z|^2 \text{ \& } |\bar{z}| = |z|]$$

$$\Rightarrow \alpha_k x_k = |\alpha_k|^q$$

--- (v)

Since $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q+p = pq \Rightarrow q = pq - p$

So by (v), we get

$$\alpha_k x_k = |\alpha_k|^{pq-p} = \frac{|\alpha_k|^{pq}}{|\alpha_k|^p} = \frac{[|\alpha_k|^q]^p}{|\alpha_k|^p} = \frac{|\alpha_k x_k|^p}{|\alpha_k|^p}$$

$$\Rightarrow \alpha_k x_k = |\alpha_k|^p \quad \text{--- (vi)} \quad [\text{by (v)}]$$

By eqn (v) & (vi), we have

$$\alpha_k x_k = |\alpha_k|^q = |x_k|^p \quad \text{--- (vii)}$$

Now $x \in l_p$

$$\text{Then } \|x\| = \left(\sum_{k=1}^n |x_k|^p \right)^{1/p}$$

$$\Rightarrow \|x\| = \left(\sum_{k=1}^n |\alpha_k|^q \right)^{1/p} \quad \text{--- (viii)} \quad [\text{by (vii)}]$$

$$\text{Also, } f(x) = \sum_{k=1}^n \alpha_k x_k = \sum_{k=1}^n |\alpha_k|^q \quad [\text{by (v)}]$$

Now we know that,

$$|f(x)| \leq \|x\| \cdot \|f\|$$

$$\Rightarrow \left| \sum_{k=1}^n |\alpha_k|^q \right| \leq \left[\sum_{k=1}^n |\alpha_k|^q \right]^{1/p} \|f\| \quad [\text{by (viii)}]$$

$$\Rightarrow \frac{\left| \sum_{k=1}^n |\alpha_k|^q \right|}{\left[\sum_{k=1}^n |\alpha_k|^q \right]^{1/p}} \leq \|f\|$$

$$\Rightarrow \left[\sum_{k=1}^n |\alpha_k|^q \right]^{1-\frac{1}{p}} \leq \|f\|$$

$$\Rightarrow \left[\sum_{k=1}^n |\alpha_k|^q \right]^{1/q} \leq \|f\|$$

$$\Rightarrow \|x\| \leq \|f\| < \infty \quad \text{--- (ix)}$$

Conversely, suppose $\alpha = \{\alpha_k\} \in l_q$ is given

Define f on l_p by

$$f(x) = \sum_k \alpha_k x_k \quad \text{where } x = \{x_k\} \in l_p$$

To prove f is linear :

$$\text{Let } x = \{x_k\} \in l_p$$

$$y = \{y_k\} \in l_p$$

and let a & b be two any scalars.

$$\begin{aligned} \text{So } f(ax+by) &= \sum_k (ax_k x_k + by_k y_k) \\ &= a \sum_k (x_k x_k) + b \sum_k (y_k y_k) \end{aligned}$$

$$\Rightarrow f(ax+by) = a \cdot f(x) + b \cdot f(y)$$

$\Rightarrow f$ is linear.

Next we prove $f \in l_p^*$ or l_q :

$$\text{We have } f(x) = \sum_k a_k x_k$$

$$\text{Then } |f(x)| = \left| \sum_k a_k x_k \right| \leq \sum_k |a_k x_k|$$

$$\Rightarrow |f(x)| \leq \left[\sum_k |a_k|^q \right]^{1/q} \cdot \left[\sum_k |x_k|^p \right]^{1/p} \quad [\text{By Holder's inequality}]$$

$$\Rightarrow |f(x)| \leq \|a\| \cdot \|x\|$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|a\|$$

Taking supremum w.r. to x ,

$$\Rightarrow \sup \left\{ \frac{|f(x)|}{\|x\|} : \|x\| \neq 0 ; x \in l_p \right\} \leq \|a\|$$

$$\Rightarrow \|f\| \leq \|a\|$$

— (x)

$$\Rightarrow f \in l_p^* \text{ or } l_q$$

So by eqn (ix) & (x), we get

$$\|f\| = \|a\|$$

Therefore the mapping $T: l_p^* \rightarrow l_q$ defined by

$$T(f) = a = \{a_k\}$$

is an isometric isomorphism.

Hence l_p^* is isometric isomorphic to l_q .

OR. Next we show that T is an isometry.

- T is one-one :

For this let $f, g \in l_p^*$ such that

$$T(f) = \alpha, \quad T(g) = \beta.$$

$$\text{and } f(x) = \sum_k \alpha_k x_k, \quad g(x) = \sum_k \beta_k x_k.$$

$$\because T(f) = T(g) \Rightarrow \alpha = \beta$$

$$\Rightarrow \alpha_k = \beta_k \quad \forall k$$

$$\Rightarrow \sum_k \alpha_k x_k = \sum_k \beta_k x_k$$

$$\Rightarrow f(x) = g(x)$$

$$\Rightarrow (f-g)(x) = 0$$

$$\Rightarrow f-g = 0$$

$$\Rightarrow T(f) = T(g) \Rightarrow f = g$$

$$\Rightarrow T \text{ is one-one.}$$

- T is onto :

For this $\alpha = \{\alpha_k\} \in l_q$, $\exists f \in l_p^*$ defined by

$$T(f) = \alpha$$

$$\Rightarrow T \text{ is onto.}$$

- T is linear :

$$T(af+bg) = (af+bg)(e_k)$$

$$= af(e_k) + bg(e_k)$$

$$= a \sum \alpha_k + b \sum \beta_k$$

$$= a\alpha + b\beta$$

$$\Rightarrow T(af+bg) = aT(f) + bT(g)$$

$$\Rightarrow T \text{ is linear.}$$

• T is norm preserving :

We have $T(f) = \alpha$

$$\|T(f)\| = \|\alpha\| = \|f\|$$

$$\Rightarrow \|T(f)\| = \|f\|$$

OR

$$\|T(f) - T(g)\| = \|T(f-g)\| = \|f-g\|$$

$$\Rightarrow \|T(f) - T(g)\| = \|f-g\|$$

$\Rightarrow T$ is norm preserving.

Thus T is an isometry.

Hence l_p^* is isometrically isomorphic to l_q .

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Theorem⁽⁸⁰⁾: l_1^* is isometrically isomorphic to l_∞ .

Proof :

Let $f \in l_1^*$

Since $\{e_1, e_2, \dots, e_n, \dots\}$ be the basis of l_1
and $x \in l_1$

$\therefore$ We can express x by

$$x = \sum_k x_k e_k \quad \text{where } x = \{x_k\} \quad \text{--- (i)}$$

Define $\alpha_k = f(e_k)$

So by linearity and continuity of f ,

$$f(x) = \sum_k x_k f(e_k) = \sum_k x_k \cdot \alpha_k \quad \text{--- (ii)}$$

Now for a given positive integer n ,
we can choose as

$$x_k = \begin{cases} \operatorname{sgn} \bar{\alpha}_n & , \text{ if } k=n \\ 0 & , \text{ if } k \neq n \end{cases} \quad \text{--- (iii)}$$

$$\text{where } \operatorname{sgn} c = \begin{cases} 0 & , \text{ if } c = 0 \\ \frac{c}{|c|} & , \text{ if } c \neq 0 \end{cases} \quad \text{--- (iv)}$$

$$\begin{aligned} \text{Then } \|x\| &= \left\| \sum_k x_k e_k \right\| && [\text{by (i)}] \\ &\leq \sum_k |x_k| \|e_k\| \leq \sum_k |x_k| = |\operatorname{sgn} \bar{\alpha}_n| && [\text{by (iii)}] \\ &= \left| \frac{\bar{\alpha}_n}{|\bar{\alpha}_n|} \right| = \frac{|\bar{\alpha}_n|}{|\bar{\alpha}_n|} = 1 \end{aligned}$$

$$\Rightarrow \|x\| \leq 1 \quad \text{--- (v)}$$

$$\begin{aligned} \text{and } f(x) &= \sum_k x_k \alpha_k = \operatorname{sgn} \bar{\alpha}_n \cdot \alpha_n && [\text{by (iii)}] \\ &= \frac{\bar{\alpha}_n}{|\bar{\alpha}_n|} \cdot \alpha_n && [\text{by (iv)}] \\ &= \frac{|\alpha_n|^2}{|\bar{\alpha}_n|} && [\because z \cdot \bar{z} = |z|^2, |\bar{z}| = |z|] \\ \Rightarrow f(x) &= |\alpha_n| \end{aligned}$$

$$\text{So, } |\alpha_n| = f(x) \leq |f(x)| \leq \|f\| \cdot \|x\|$$

$$\Rightarrow |\alpha_n| \leq \|f\| \quad [\text{by (v) ; } \|x\| \leq 1]$$

Taking supremum,

$$\Rightarrow \sup_n |\alpha_n| \leq \|f\|$$

$$\Rightarrow \|x\| \leq \|f\| \quad \text{--- (vi)}$$

Conversely, suppose that $\alpha = \{\alpha_k\} \in \ell_\infty$ is given
Define f on ℓ_1 by

$$f(x) = \sum_k x_k \alpha_k \quad \text{where } x = \{x_k\} \in \ell_1$$

To prove f is linear :

Let $x = \{x_k\} \in \ell_1$ and $y = \{y_k\} \in \ell_1$
and let a & b are two scalars.

$$\begin{aligned} \text{Then } f(ax+by) &= \sum_k (ax_k \alpha_k + by_k \alpha_k) \\ &= a \sum_k x_k \alpha_k + b \sum_k y_k \alpha_k \end{aligned}$$

$$\Rightarrow f(ax+by) = a f(x) + b f(y)$$

$\Rightarrow f$ is linear.

Next we prove $f \in l_1^*$ or l_∞ :

$$\text{We have } f(x) = \sum_k \alpha_k x_k$$

$$\therefore |f(x)| = \left| \sum_k \alpha_k x_k \right| \leq \sum_k |\alpha_k x_k| \leq \left[\sum_k |\alpha_k| \right] \sup_k |x_k|$$

$$\Rightarrow |f(x)| \leq \|x\| \cdot \|f\|$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|f\|$$

Taking supremum w.r.to x

$$\Rightarrow \sup \left\{ \frac{|f(x)|}{\|x\|} : \|x\| \neq 0 ; x \in l_1 \right\} \leq \|f\|$$

$$\Rightarrow \|f\| \leq \|f\| \quad \text{--- (vii)}$$

$$\Rightarrow f \in l_1^* \text{ or } l_\infty$$

By eqn (vi) and (vii), we get

$$\|f\| = \|f\|$$

Thus the mapping $T: l_1^* \rightarrow l_\infty$ defined by

$$T(f) = \alpha = \{\alpha_k\}$$

is an isometric isomorphism.

OR Next we show that T is an isometry.

• T is one-one :

For this let $f, g \in l_1^*$ such that

$$T(f) = \alpha, \quad T(g) = \beta$$

$$\text{and } f(x) = \sum_k \alpha_k x_k, \quad g(x) = \sum_k \beta_k x_k$$

$$\therefore T(f) = T(g) \Rightarrow \alpha = \beta \Rightarrow \alpha_k = \beta_k \quad \forall k$$

$$\Rightarrow \sum_k \alpha_k x_k = \sum_k \beta_k x_k$$

$$\Rightarrow T(f) = T(g) \Rightarrow f(x) = g(x) \\ \Rightarrow (f-g)(x) = 0$$

$$\Rightarrow T(f) = T(g) \Rightarrow f = g$$

$\Rightarrow T$ is one-one.

• T is onto :

For $\alpha = \{\alpha_k\} \in \ell_\infty$, $\exists f \in \ell_1^*$ defined by

$$T(f) = \alpha$$

$\Rightarrow T$ is onto.

• T is linear :

Now,

$$T(af+bg) = (af+bg)(e_k) = af(e_k) + bg(e_k) \\ = a\alpha + b\beta$$

$$\Rightarrow T(af+bg) = aT(f) + bT(g)$$

$\Rightarrow T$ is linear.

• T is norm preserving :

$$\text{Since } T(f) = \alpha$$

$$\therefore \|T(f)\| = \|\alpha\| = \|f\|$$

$$\Rightarrow \|T(f)\| = \|f\|$$

$\Rightarrow T$ is norm preserving.

Thus T is an isometry.

Hence ℓ_1^* is isometrically isomorphic to ℓ_∞ .

Theorem ⁽⁸¹⁾: c^* is isometrically isomorphic to ℓ_1 .

Proof: Let $f \in c^*$

Since $\{e_1, e_2, \dots, e_n, \dots\}$ is a basis for C ,
and let $x = \{a_n\} \in C$.

Then we can express x by

$$x = 1e + \sum_{k=1}^{\infty} (x_k - 1) e_k \quad \text{--- (i)}$$

where $x = \{x_k\}$ and $1 = \lim_{n \rightarrow \infty} x_n$

By linearity and continuity of f , we have

$$f(x) = 1 \cdot f(e) + \sum_{k=1}^{\infty} (x_k - 1) \cdot f(e_k) \quad \text{--- (ii)}$$

For a given positive integer n ,

$$\text{let } x_k = \begin{cases} \operatorname{sgn} f(e_k) & ; 1 \leq k \leq n \\ 0 & ; k > n \end{cases} \quad \text{--- (iii)}$$

where

$$\operatorname{sgn} c = \begin{cases} 0 & ; \text{if } c = 0 \\ \frac{c}{|c|} & ; \text{if } c \neq 0 \end{cases} \quad \text{--- (iv)}$$

Let $x \in C_0$

then $\lim_{n \rightarrow \infty} x_n = 0$

$$\therefore \|x\| = \sup_k |x_k| = 1$$

$$\Rightarrow \|x\| = 1 \quad \text{--- (v)}$$

$$\times \{ \text{And } |f(x)| = |1f(e) + \sum_{k=1}^{\infty} (x_k - 1) \cdot f(e_k)|$$

$$= |0 + \sum_{k=1}^{\infty} (x_k - 0) f(e_k)|$$

$$= \left| \sum_{k=1}^{\infty} \operatorname{sgn} f(e_k) \cdot f(e_k) \right| \quad \text{[by (iii)]}$$

$$= \left| \sum_{k=1}^{\infty} \frac{f(e_k)}{|f(e_k)|} \cdot f(e_k) \right| \quad \text{[by (iv)]}$$

$$\leq \sum_{k=1}^{\infty} |f(e_k)| \leq \|f\| \cdot \|x\| = \|f\| \quad \text{[by (v)]}$$

$$\Rightarrow |f(x)| \leq \sum_{k=1}^{\infty} |f(e_k)| \leq \|f\|$$

--- (vi) }

From eqn (ii), we have

$$\begin{aligned}
 f(x) &= 1 \cdot f(e) + \sum_k (x_k - 1) \cdot f(e_k) \\
 &= 1 \cdot f(e) + \sum_k x_k \cdot f(e_k) - \sum_k 1 \cdot f(e_k) \\
 &= 1 [f(e) - \sum_k f(e_k)] + \sum_k f(e_k) \cdot x_k \\
 \Rightarrow f(x) &= a + \sum_k a_k x_k \quad \text{--- (vii)}
 \end{aligned}$$

where,

$$a = f(e) - \sum_k f(e_k) \quad \text{and} \quad a_k = f(e_k)$$

Now, we define

$$x_k = \begin{cases} \operatorname{sgn} a_k & ; 1 \leq k \leq n \\ \operatorname{sgn} a & ; k > n \end{cases} \quad \text{--- (viii)}$$

$$\text{Let } 1 = \lim_{n \rightarrow \infty} x_n = \operatorname{sgn} b$$

Then by (vii),

$$\begin{aligned}
 |f(x)| &= |a + \sum_k a_k x_k| \\
 &= |\operatorname{sgn} b \cdot a + \sum_k a_k x_k| \\
 &= \left| \frac{b}{|b|} \cdot a + \sum_k a_k x_k \right| \quad [\text{by (iv), (viii)}] \\
 &= \left| a + \sum_{k=1}^n a_k \cdot \operatorname{sgn} a_k + \sum_{k=n+1}^{\infty} a_k \operatorname{sgn} a \right| \\
 &= \left| a + \sum_{k=1}^n a_k \cdot \frac{a_k}{|a_k|} + \sum_{k=n+1}^{\infty} a_k \cdot \operatorname{sgn} a \right| \\
 &= \left| a + \sum_{k=1}^n a_k + \sum_{k=n+1}^{\infty} a_k \operatorname{sgn} a \right| \quad [\text{by (iv)}] \\
 &= \left| a + \sum_{k=1}^n a_k + 0 \right| \quad [\because a = \{a_n\} \in l_1] \\
 &= \left| a + \sum_{k=1}^n a_k \right| \quad \because \sum_{k=n+1}^{\infty} a_k \rightarrow 0 \text{ as } n \rightarrow \infty] \\
 &= \left| a + \sum_{k=1}^n a_k \right| \leq \|f\| \cdot \|x\| = \|f\| \\
 \Rightarrow |f(x)| &= \left| a + \sum_{k=1}^n a_k \right| \leq \|f\| \quad \text{--- (ix)}
 \end{aligned}$$

$$\text{Take } |\lim x_n| \leq \|x\|$$

Then by eqn (vii),

$$\begin{aligned}
 |f(x)| &= |a + \sum_k a_k x_k| \\
 &= |\lim x_n \cdot a + \sum_k a_k \cdot x_k| \\
 &\leq |\|x\| \cdot a + \sum_k a_k \cdot \|x\||
 \end{aligned}$$

$$\Rightarrow |f(x)| \leq |a + \sum_k a_k| \|x\|$$

$$\Rightarrow |f(x)| \leq |a + \sum_k a_k| \cdot \|x\|$$

$$\Rightarrow \frac{|f(x)|}{\|x\|} \leq |a + \sum_k a_k|$$

Taking supremum w.r. to x

$$\Rightarrow \sup \left\{ \frac{|f(x)|}{\|x\|} : \|x\| \neq 0 ; x \in L \right\} \leq |a + \sum_k a_k|$$

$$\Rightarrow \|f\| \leq |a + \sum_k a_k| \quad \text{--- (x)}$$

So by eqn (ix) and (x),

$$\|f\| = |a + \sum_k a_k|$$

Therefore the mapping $T: c^* \rightarrow L$ be defined by

$$T(f) = \{a, a_1, \dots\}$$

is an isometric isomorphism.

Hence c^* is isometrically isomorphic to L .

18-19/2.5

20-21/2.5

Theorem: Let X and Y are normed linear spaces and let T is linear transformation from X into Y . Then T is continuous at every point of X or at no point of X .

Proof:

Suppose X and Y are normed linear spaces and let $T: X \rightarrow Y$ be linear transformation.

Let $\dim X = n$

and suppose $B = \{e_1, e_2, \dots, e_n\}$ be basis of X .
Then for $x \in X$

$$x = \sum_{i=1}^n \alpha_i e_i$$

Since $T: X \rightarrow Y$ is linear transformation

$$\Rightarrow T(x) = T\left(\sum_{i=1}^n \alpha_i e_i\right)$$

$$= \sum_{i=1}^n \alpha_i T(e_i) \quad \text{--- (i)}$$

We know that in a finite dimensional nls all the norms are equivalent.

Let the norm on X is defined as

$$\|x\|_0 = \max_i |\alpha_i|$$

$$\Rightarrow \|x\|_0 \geq |\alpha_i| \quad \text{--- (ii)}$$

Also, let the norm on Y is defined by $\|\cdot\|$ s.t.

$$\|Tx\| = \left\| \sum_{i=1}^n \alpha_i T(e_i) \right\| \quad \text{[by (i)]}$$

$$\leq \sum_{i=1}^n \|\alpha_i T(e_i)\|$$

$$\leq \sum_{i=1}^n |\alpha_i| \|T(e_i)\|$$

$$\Rightarrow \|Tx\| \leq \|x\|_0 \cdot \sum_{i=1}^n \|T(e_i)\| \quad \text{--- (iii) [by eqn (ii)]}$$

Since $B = \{e_1, e_2, \dots, e_n\}$ be basis of X .

$$\Rightarrow \exists \text{ a positive constant } M > 0 \text{ such that}$$

$$\sum_{i=1}^n \|T(e_i)\| = M > 0 \quad \text{--- (iv)}$$

From eqn (iii) and (iv),

$$\|Tx\| \leq M \cdot \|x\|_0$$

$$\Rightarrow T \text{ is bounded.}$$

So by the result, "If X & Y are nls and $T: X \rightarrow Y$ be linear transformation & bounded, Then T is continuous at every point of X ".

$\Rightarrow T$ is continuous at every point of X .